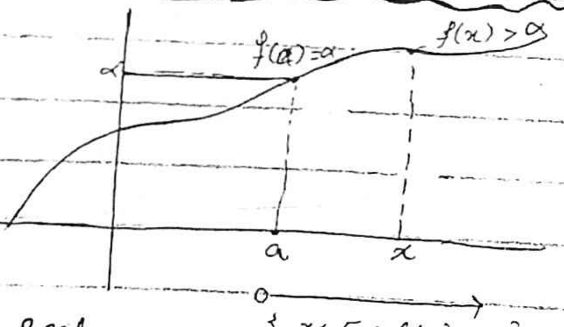
Chapter 3

Measurable Functions

Lebesgue Measurable Functions:

Let $(X, \mathcal{A}, \mu)$ be a measure space and $E \in \mathcal{A}$. An extended real valued function f on E is said to be measurable function if for each $\alpha \in \mathbb{R}$, $\{x \in E : f(x) > \alpha\} \in \mathcal{A}$. In particular $\mathcal{A}$ is the class M of Lebesgue measurable subsets of $\mathbb{R}$, then the measurable function 'f' is called a Lebesgue measurable function

> f is Measurable $\Leftrightarrow$ $\{x \in E : f(x) > \alpha\}$ is measurable for each $\alpha \in \mathbb{R}$



$\{x \in E : f(x) > \alpha\}$

Theorem: Imp A | 2013, 2017

Let 'f' be an extended real valued function defined on a measurable set D. Then the following are equivalent.

(i) f is measurable

(ii) $\{x : f(x) \geq \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$

(iii) $\{x : f(x) < \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$

(iv) $\{x : f(x) \leq \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$

Moreover statement (i) → (iv) imply that,

(v) $\{x : f(x) = \alpha\}$ is a measurable set for each extended real number α. ----- ---

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$\bigcap_{n=1}^{\infty} \left(a-\frac{1}{n}, b+\frac{1}{n}\right) =$

$\bigcap_{n=1}^{\infty} \left(a-\frac{1}{n}, b\right) =$

Proof:-

(i) ⇒ (ii) Let 'f' is measurable then $\{x \in D : f(x) > \alpha\}$ is measurable

We prove (ii) by showing that

$$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) > \alpha - \frac{1}{n}\right\}$$

For this

Let $y \in \{x : f(x) \geq \alpha\}$

so that $f(y) \geq \alpha \quad \forall \alpha \in \mathbb{R}$

In particular $f(y) > \alpha - \frac{1}{n}$ for each $n \in N$

⇒

$y \in \left\{x : f(x) > \alpha - \frac{1}{n}\right\}$ for each $n \in N$

Hence

$$y \in \bigcap_{n=1}^{\infty} \left\{x : f(x) > \alpha - \frac{1}{n}\right\}$$

$$\Rightarrow \{x : f(x) \geq \alpha\} \subseteq \bigcap_{n=1}^{\infty} \left\{x : f(x) > \alpha - \frac{1}{n}\right\} \quad \text{— (i)}$$

For converse inclusion suppose that

$t \in \bigcap_{n=1}^{\infty} \left\{x : f(x) > \alpha - \frac{1}{n}\right\}$

then

$t \in \left\{x : f(x) > \alpha - \frac{1}{n}\right\}$ for all $n \in N$ & so

$f(t) > \alpha - \frac{1}{n}$ for all $n \in N$

Thus $f(t) \geq \alpha$ for all $\alpha \in \mathbb{R}$

$\Rightarrow t \in \{x : f(x) \geq \alpha\} \quad \forall \alpha \in \mathbb{R}$

$$\Rightarrow \bigcap_{n=1}^{\infty} \left\{x : f(x) > \alpha - \frac{1}{n}\right\} \subseteq \{x : f(x) \geq \alpha\} \quad \text{— (ii)}$$

Hence (i) and (ii) ⇒

$$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) > \alpha - \frac{1}{n}\right\}$$

Since $\{x : f(x) > \alpha - 1/n\}$ is a measurable set

The intersection of countable number of measurable sets is a measurable set and so $\{x : f(x) \geq \alpha\}$ is a measurable set.

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(ii) ⇒ (iii) Suppose that $\{x : f(x) \geq \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$.
To show that $\{x : f(x) < \alpha\}$ is measurable, we will show that

$$\{x : f(x) < \alpha\} = D \setminus \{x : f(x) \geq \alpha\}$$

Let $y \in \{x \in D : f(x) < \alpha\}$

$\Rightarrow f(y) < \alpha$

$\Rightarrow y \in D$ and $y \notin \{x \in D : f(x) \geq \alpha\}$

$\Rightarrow y \in D \setminus \{x : f(x) \geq \alpha\}$

$\Rightarrow \{x \in D : f(x) < \alpha\} \subseteq D \setminus \{x \in D : f(x) \geq \alpha\}$ — (i)

For the other inclusion,

let $t \in D \setminus \{x \in D : f(x) \geq \alpha\}$

$\Rightarrow t \in D$ but $t \notin \{x \in D : f(x) \geq \alpha\}$

$\Rightarrow t \in \{x \in D : f(x) < \alpha\}$

$\Rightarrow D \setminus \{x \in D : f(x) \geq \alpha\} \subseteq \{x \in D : f(x) < \alpha\}$ — (ii)

from (i) and (ii)

$\{x \in D : f(x) < \alpha\} = D \setminus \{x \in D : f(x) \geq \alpha\}$

As D is measurable and from (ii)

$\{x \in D : f(x) \geq \alpha\}$ is also measurable

so $D \setminus \{x \in D : f(x) \geq \alpha\}$ is measurable

And so

$\{x \in D : f(x) < \alpha\}$ is Measurable.

(iii) ⇒ (iv)

Suppose that (iii) holds i.e

$\{x \in D : f(x) < \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$

To show that $\{x : f(x) \leq \alpha\}$ is measurable

we show that $\{x \in D : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in D : f(x) < \alpha + \frac{1}{n}\}$

Let $y \in \{x : f(x) \leq \alpha\}$ so that
$f(y) \leq \alpha \;\; \forall \; \alpha \in \mathbb{R}$
In particular $f(y) < \alpha + 1/n$ for each $n \in N$
This implies that $y \in \{x : f(x) < \alpha + 1/n\}$ for each $n \in N$

$$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\}$$

$$\Rightarrow \{x : f(x) \leq \alpha\} \subseteq \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\} \quad \text{---(ii)}$$

Now for converse inclusion
Let $t \in \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\}$

$\Rightarrow t \in \{x : f(x) < \alpha + 1/n\}$ for all $n \in N$ and so
$f(t) < \alpha + \frac{1}{n}$ for all $n \in N$
Thus $f(t) \leq \alpha \;\; \forall \; \alpha \in \mathbb{R}$
$\Rightarrow t \in \{x : f(x) \leq \alpha\}$

$$\Rightarrow \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\} \subseteq \{x : f(x) \leq \alpha\} \quad \text{---(ii)}$$

Now from (i) and (ii)
$$\{x : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\}$$

As $\bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\}$ being intersection of measurable sets is measurable from (iii)
Thus
$\{x : f(x) \leq \alpha\}$ is measurable.

(iv) ⇒ (i) We show that
$$\{x : f(x) > \alpha\} = D \setminus \{x \in D : f(x) \leq \alpha\}$$
Let $y \in \{x : f(x) > \alpha\}$
then $f(y) > \alpha$
$y \in D$ and $y \notin \{x : f(x) \leq \alpha\}$
$\Rightarrow y \in D \setminus \{x : f(x) \leq \alpha\}$
$$\Rightarrow \{x : f(x) > \alpha\} \subseteq D \setminus \{x : f(x) \leq \alpha\} \quad \text{---(i)}$$

Now conversely suppose that

$t \in D \setminus \{x : f(x) \leq \alpha\}$

$\Rightarrow t \in D$ and $t \notin \{x \in D : f(x) \leq \alpha\}$

$\Rightarrow t \in \{x \in D : f(x) > \alpha\}$

$\Rightarrow$

$D \setminus \{x : f(x) \leq \alpha\} = \{x \in D : f(x) > \alpha\}$

As D is measurable and from (iv) $\{x : f(x) \leq \alpha\}$ is measurable so $D \setminus \{x : f(x) \leq \alpha\}$ is measurable and So $\{x : f(x) > \alpha\}$ is Measurable. ✓

(v)

Let $\alpha \in \mathbb{R}$ and $\{x \in D : f(x) = \alpha\} = \{x \in D : f(x) \geq \alpha\} \cap \{x \in D : f(x) \leq \alpha\}$ being intersection of measurable sets is measurable set.

When $\alpha = \infty$ we have

$$\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) > n\}$$

and.

if $\{\alpha = -\infty\}$ then

$$\{x : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) < -n\}$$

Consequently $\{x : f(x) = \alpha\}$ is measurable set for extended real number α. *

24-04-19

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* **Definition 1.5:**

Let 'f' be an extended real valued function defined on any set A. The positive part f^+ and the negative part f^- of f are the extended real valued functions defined by:

$f^+(x) = \max\{f(x), 0\} = f \vee 0$

and $f^-(x) = \max\{-f(x), 0\} = -f \vee 0$ for all $x \in A$.

✓ (i) Show that $f = f^+ - f^-$

For this let $x \in A$ then there are three possibilities for the values of 'f' at 'x'.

(a) If $f(x) = 0$ then $f^+(x) = \max\{f(x), 0\} = \max\{0,0\} = 0$

and $f^-(x) = \max\{-f(x), 0\} = \max\{0, 0\} = 0$

This implies

$(f^+ - f^-)(x) = f^+(x) - f^-(x) = 0 - 0 = 0 = f(x) \; \forall x \in A$.

(b) If $f(x) > 0$ then $f^+(x) = \max\{f(x), 0\} = f(x)$

and $f^-(x) = \max\{-f(x), 0\} = 0$

Then $(f^+ - f^-)(x) = f^+(x) - f^-(x) = f(x) + 0 = f(x) \; \forall x \in A$

(c) If $f(x) < 0$ then $f^+(x) = \max\{f(x), 0\} = 0$

and $f^-(x) = \max\{-f(x), 0\} = -f(x)$

Thus $(f^+ - f^-)(x) = f^+(x) - f^-(x) = 0 - (-f(x))$

$= f(x) \; \forall \; x \in A$

All cases shows that $f = f^+ - f^-$

✓ (ii) Prove that $|f| = f^+ + f^-$

For this let $x \in A$ then there are three possibilities

(a) If $f(x) = 0 \; \forall x \in A$ then $|f|(x) = |f(x)| = 0 \; \forall x \in A$

and $f^+(x) = \max\{f(x), 0\} = 0$ and $f^-(x) = \max\{-f(x), 0\} = 0$

Then $|f| = 0 = f^+ + f^-$.

(b) If $f(x) < 0$ $\forall x \in A$ then $|f| = -f$
and $f^+(x) = \text{Max}\{f(x), 0\} = 0$
and $f^-(x) = \text{Max}\{-f(x), 0\} = -f(x)$

then $f^+ + f^- = 0 - f = -f = |f|$ ✓ x

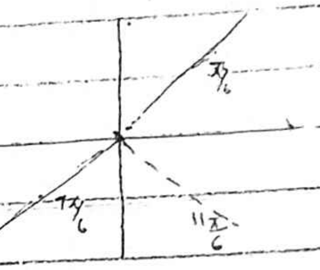
(c) If $f(x) > 0$ $\forall x \in A$ then $|f| = f$
and $f^+(x) = \text{Max}\{f(x), 0\} = f(x)$
and $f^{-1}(x) = \text{Max}\{-f(x), 0\} = 0$
Then $f^+ + f^{-1} = f + 0 = f = |f|$

$\frac{180}{6} = 30$

$\sin 30 = \frac{1}{2}$

Hence all three cases show
$|f| = f^+ + f^-$ ✓

Example:- If $f(x) = \frac{1}{2} + \sin x$, $0 < x < 2\pi$
Then $f(x) = 0$
$\Rightarrow \frac{1}{2} + \sin x = 0$
$\Rightarrow \sin x = -1/2$
$x = 7\pi/6, 11\pi/6$



$2\pi - \pi/6$

$f^+ = \max\{f(x), 0\}$
We have three possible intervals.
$[0, 7\pi/6)$ $[7\pi/6, 11\pi/6]$ and $(11\pi/6, 2\pi)$

$\frac{7\pi}{6} = 210$

At $x = 0$
$f^+ = \max\{\frac{1}{2} + \sin x, 0\} = \frac{1}{2} + \sin x$
At $x = \frac{\pi}{2}$ $f^+ = \max\{\frac{1}{2} + \sin x, 0\} = \frac{1}{2} + \sin x$

$\Rightarrow f^+ = \frac{1}{2} + \sin x$ when $0 < x < 7\pi/6$
when $7\pi/6 \le x \le 11\pi/6$
At $x = 7\pi/6$

$\sin \frac{7\pi}{6} = 0$

$f^+ = \max\{\frac{1}{2} + \sin x, 0\} = 0$ ✓

when $x = 3\pi/2$

$f^+ = \max\{1/2 + \sin x, 0\} = 0$ ✓

when $x = 11\pi/6$

$f^+ = \max\{1/2 + \sin x, 0\} = 0$ ✓

$\Rightarrow f^+ = 0 \quad 7\pi/6 \le x \le 11\pi/6$

$11\pi/6 < x < 2\pi$

when $x = 23\pi/12$

$f^+ = \max\{1/2 + \sin x, 0\} = 1/2 + \sin x$

Thus $f^+ = \begin{cases} \frac{1}{2} + \sin x & 0 \le x < 7\pi/6 \\ 0 & 7\pi/6 \le x \le 11\pi/6 \\ 1/2 + \sin x & 11\pi/6 < x < 2\pi \end{cases}$

When $x = 0$

$f^- = \max\{-f(x), 0\} = 0$

when $x = \pi/2$

$f^- = \max\{-f(x), 0\} = 0$

when $x = 7\pi/6$

$f^- = \max\{-f(x), 0\} = -\frac{1}{2} - \sin x = -f(x)$

$\Rightarrow f^- = 0$ for $0 \le x < 7\pi/6$

When $x = 3\pi/2$

$f^- = \max\{-f(x), 0\} = -1/2 - \sin x = -f(x)$

when $x = 11\pi/6$ $f^- = \max\{-f(x), 0\} = -\frac{1}{2} - \sin x = -f(x)$

$\Rightarrow f^- = -\frac{1}{2} - \sin x$ for $\frac{7\pi}{6} \le x \le 11\pi/6$

when $x = 23\pi/12$

$f^- = \max\{-f(x), 0\} = 0$

$\frac{2\pi - \frac{11\pi}{6}}{2} = \frac{\pi/6}{2} \quad \frac{\frac{12\pi - 11\pi}{6}}{2} = \frac{\pi/6}{2} = \pi/12$

Thus

$$f^- = \begin{cases} 0 & 0 \le x < 7\pi/6 \\ -\frac{1}{2} - \sin x & 7\pi/6 \le x \le 11\pi/6 \\ 0 & 11\pi/6 < x < 2\pi \end{cases}$$

Clearly $f = f^+ - f^-$
and $|f| = f^+ + f^-$

Theorem 1.6 Imp A|2011
Let 'f' and 'g' be two real valued measurable functions defined on the same measurable domain D and 'c' be any real number. Then the functions (i) $f+c$ (ii) cf (iii) $f+g$ (iv) $f-g$ (v) f^2 (vi) fg (vii) $\frac{f}{g}$ $(g \neq 0)$ (viii) $f \vee g$ (ix) $f \wedge g$ (x) $|f|$ are measurable functions.

Proof: (i) $f+c$
Let f be a measurable function and 'c' be any real number.
Consider $\{x : (f+c)(x) > \alpha\} = \{x : f(x) + c > \alpha\} = \{x : f(x) > \alpha - c\}$ for each $\alpha \in \mathbb{R}$
Now since 'f' is measurable so let $\alpha - c = \alpha'$
then $\{x : f(x) > \alpha' ; \alpha' = \alpha - c\}$ is measurable.
This implies $\{x : (f+c)(x) > \alpha\}$ is measurable
$\Rightarrow$ $f+c$ is measurable.

(ii) cf
To show that cf is a measurable function we shall consider the following three cases:
1) $c = 0$
If $\alpha \le 0$ then

$\{x \in D : (cf)(x) > \alpha\} = \{x \in D : cf(x) > \alpha\} = D$

Since D is measurable so

$\{x \in D : (cf)(x) > \alpha\}$ is measurable and

so cf is measurable

If $\alpha > 0$

then

$\{x \in D : (cf)(x) > \alpha\} = \{x \in D : cf(x) > \alpha\} = \phi$

ϕ is measurable $\Rightarrow \{x \in D : (cf)(x) > \alpha\}$ is measurable

$\Rightarrow$ cf is measurable.

2 $c < 0$

The set $\{x \in D : (cf)(x) > \alpha\} = \{x \in D : c f(x) > \alpha\}$

$= \{x \in D : f(x) < \frac{\alpha}{c}\}$ which is a measurable set

($\because$ 'f' is measurable)

3 $c > 0$

The set $\{x \in D : (c.f)(x) > \alpha\} = \{x \in D : cf(x) > \alpha\}$

$= \{x \in D : f(x) > \frac{\alpha}{c}\}$

ie for $\alpha' = \frac{\alpha}{c}$

$\{x \in D : f(x) > \alpha'\}$ which is measurable

Since 'f' is measurable.

(iii) If $f(x) + g(x) > \alpha$ then $f(x) > \alpha - g(x)$ $\forall$ all $\alpha \in \mathbb{R}$ then there exist a rational number 'r' such that $f(x) > r > \alpha - g(x)$. Now we will show that

$\{x : (f+g)(x) > \alpha\} = \bigcup_{r \in Q} \{x : f(x) > r\} \cap \{x : \alpha - g(x) < r\}$

For this set $y \in \{x : (f+g)(x) > \alpha\}$

This implies $f(y) + g(y) > \alpha$ and so

$f(y) > \alpha - g(y)$

By above argument there exist a rational number $r \in Q$ such that

$f(y) > r > \alpha - g(y)$.

This implies that

$y \in \{x : f(x) > r\}$ and $y \in \{x : \alpha - g(y) < r\}$

Hence

$y \in \{x : f(x) > r\} \cap \{x : \alpha - g(x) < r\}$

which gives that

$y \in \bigcup_{r \in Q} \left[\{x : f(x) > r\} \cap \{x : \alpha - g(x) < r\}\right]$

$\{x : (f+g)(x) > \alpha\} \subseteq \bigcup_{r \in Q} \left[\{x : f(x) > r\} \cap \{x : \alpha - g(x) < r\}\right]$

Conversely, suppose that

$y \in \bigcup_{r \in Q} \left[\{x : f(x) > r\} \cap \{x : \alpha - g(x) < r\}\right]$

This implies there exist $r \in Q$ such that

$\Rightarrow y \in \{x : f(x) > r\} \cap \{x : \alpha - g(x) < r\}$

So $y \in \{x : f(x) > r\}$ and $y \in \{x : \alpha - g(x) < r\}$

$\Rightarrow f(y) > r$ and $\alpha - g(y) < r$

$\Rightarrow f(y) > r > \alpha - g(y)$

In particular

$f(y) > \alpha - g(y)$

$\Rightarrow f(y) + g(y) > \alpha = (f+g)(y) > \alpha$

Thus $y \in \{x : (f+g)(x) > \alpha\}$

$\Rightarrow \bigcup_{r \in Q} \left[\{x : f(x) > r\} \cap \{x : \alpha - g(x) < r\}\right] \subseteq \{x : (f+g)(x) > \alpha\}$

$\{x : (f+g)(x) > \alpha\} = \bigcup_{r \in Q} \left[\{x : f(x) > r\} \cap \{x : \alpha - g(x) < r\}\right]$

As $\{x : f(x) > r\}$ and $\{x : \alpha - g(x) < r\}$ are measurable also their intersection is measurable

and so

$\{x : (f+g)(x) > \alpha\}$ being countable union of measurable sets is measurable. ✓ (#)

• There exist countable infinite no. of rational no. b/w $f(x)$ & $\alpha - g(x)$ that's why we can take countable union

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(iv) $g-f$

As we have shown that
$cf(x)$ is measurable for $c<0$
Here $c=-1$
Then $-f$ is a measurable function
and $\because$ sum of two measurable functions is measurable so
$g+(-f) = g-f$ is measurable.

(v) f^2

If $\alpha < 0$ Then
$\{x : f^2(x) > \alpha\} = D$
As D is measurable so f^2 is measurable.

If $\alpha \geq 0$ Then
$\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}$

$f^2 > \alpha$
$\Rightarrow f > \sqrt{\alpha}$
and $f < -\sqrt{\alpha}$

As $\{x : f(x) > \sqrt{\alpha}\}$ is measurable ($\because f$ is measurable)
Also
$\{x : f(x) < -\sqrt{\alpha}\}$ is measurable ($\because f$ is measurable)

Then $\{x : f^2(x) > \alpha\}$ being union of two measurable sets is measurable.
$\Rightarrow f^2$ is measurable.

(vi) fg

Clearly
$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$

$(f+g)^2 = f^2 + g^2 + 2fg \rightarrow (1)$
$(f-g)^2 = f^2 + g^2 - 2fg \rightarrow (2)$
$(1) - (2)$
$(f+g)^2 - (f-g)^2 = 4fg$
$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$

As $f+g$ and $f-g$ are measurable sets
$(f+g)^2 - (f-g)^2$ is also measurable
And so $\frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ is measurable
So fg is measurable

(vii) $\frac{f}{g}$

We note that $g(x) \neq 0$

If $\alpha = 0$

$\{x \in D : \frac{1}{g(x)} > \alpha\} = \{x \in D : g(x) > 0\}$

If $\alpha > 0$

then $\{x \in D : \frac{1}{g(x)} > \alpha\} = \{x : g(x) > 0\} \cap \{x : g(x) < \frac{1}{\alpha}\}$

and if $\alpha < 0$

$\{x \in D : \frac{1}{g(x)} > \alpha\} = \{x : g(x) > 0\} \cup [\{x : g(x) < 0\} \cap \{x : g(x) < \frac{1}{\alpha}\}]$

which implies

$\frac{1}{g}$ is a measurable function.

and

so $f \cdot \frac{1}{g} = \frac{f}{g}$ $(g \neq 0)$ is measurable function.

(viii) $f \vee g$

for any $\alpha \in \mathbb{R}$

$\{x : (f \vee g)(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\}$

As 'f' and 'g' are measurable so

$\{x : f(x) > \alpha\}$ and $\{x : g(x) > \alpha\}$ are measurable and so

$\{x : (f \vee g)(x) > \alpha\}$ being union of two measurable sets is measurable.

(ix) $f \wedge g$

For any $\alpha \in \mathbb{R}$

$\{x : (f \wedge g)(x) > \alpha\} = \{x : f(x) > \alpha\} \cap \{x : g(x) > \alpha\}$

As 'f' and 'g' are measurable so

$\{x : f(x) > \alpha\}$ and $\{x : g(x) > \alpha\}$ are measurable

and $\{x : (f \wedge g)(x) > \alpha\}$ being intersection of two measurable sets is measurable.

$\Rightarrow f \wedge g$ is Measurable.

(x) $|f|$

As $|f| = f^+ + f^-$

$f^+ = \max\{f(x), 0\} = f \vee 0$

and $f^- = \max\{-f(x), 0\} = (-f) \vee 0$

As f and $-f$ are measurable

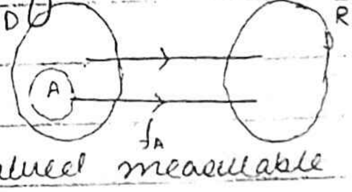
and As in (viii) $f \vee g$ is measurable when f & g are measurable

So $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ are measurable and

$|f| = f^+ + f^-$ being sum of two measurable functions is a measurable function.

THEOREM 8 Imp A/2011

Let 'f' be an extended real valued measurable function defined on D. Let A be a measurable subset of D. Then restriction of 'f' to A is also measurable.



Proof:-

Given that $f: D \to \bar{R}$ is real valued measurable function.

Let A be a measurable subset of D

we need to show that

$f_A: A \to \bar{R}$ is measurable.

As $f: D \to \bar{R}$ is measurable

So $\{x \in D : f(x) > \alpha\}$ is measurable for every $\alpha \in \mathbb{R}$

Consider $\{x \in A : f(x) > \alpha\} \subseteq \{x \in D : f(x) > \alpha\}$ for any $\alpha \in \mathbb{R}$

So $\{x \in A : f(x) > \alpha\} = A \cap \{x \in D : f(x) > \alpha\}$

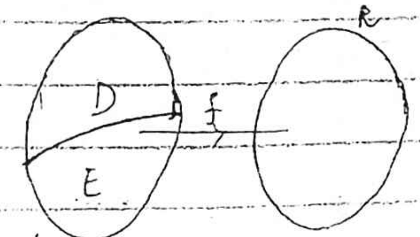
if $A \subseteq B$
$A \cap B = A$

$\Rightarrow \{x \in A : f(x) > \alpha\}$ being intersection of two measurable sets is measurable

*Corollary 1.8 :-

Let D and E be measurable sets and 'f' a function with domain $D \cup E$. Then 'f' is measurable if and only if its restriction to D and E are measurable.

Proof:-



First suppose that 'f' is measurable on D and E

That is $f_D : D \to R$ and $f_E : E \to R$ are measurable

$\{x \in D \cup E : f(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$

As $f_D : D \to R$ and $f_E : E \to R$ are measurable so the sets

$\{x \in D : f(x) > \alpha\}$ and $\{x \in E : f(x) > \alpha\}$ are measurable

So $\{x \in D \cup E : f(x) > \alpha\}$ being union of two measurable sets is measurable. Then 'f' is measurable on $D \cup E$

Conversely, suppose that 'f' is measurable on $D \cup E$
i.e. $\{x \in D \cup E : f(x) > \alpha\}$ is measurable

Then $\{x \in D : f(x) > \alpha\} \subseteq \{x \in D \cup E : f(x) > \alpha\}$ for any $\alpha \in R$

and so $\{x \in D : f(x) > \alpha\} = D \cap \{x \in D \cup E : f(x) > \alpha\}$ being intersection of two measurable sets is measurable and so restriction of f to D is measurable

Similarly restriction of f to E is measurable

24-04-19

Theorem 1.9 :-

Let 'f' be a function with measurable domain D. Then 'f' is measurable if and only if the function

$$g(x) = \begin{cases} f(x) & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases}$$

is measurable.

$A \subseteq B$
$A \cap B = A$

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* **Proof:-**

Suppose 'f' is measurable function on domain D.

and
$$g(x) = \begin{cases} f(x) & \text{if } x \in D \\ 0 & \text{if } x \notin D,\ x \in D' \end{cases}$$

Case I:-

For $x \in D$. Then $g(x) = f(x)$. As 'f' is measurable on D thus 'g' is measurable on D

Case II:-

For $x \notin D$. Then $g(x) = 0$ a constant function.
($\because$ A constant function is always measurable)

So g is measurable.

Conversely, suppose g be a measurable function on $D \cup D'$. Then

$\{x \in D : f(x) > \alpha\} \subseteq \{x \in D \cup D' : g(x) > \alpha\}$ for any $\alpha \in R$

$\Rightarrow \{x \in D : f(x) > \alpha\} = D \cap \{x \in D \cup D' : g(x) > \alpha\}$

being intersection of two measurable sets is measurable.

Theorem 1.10:

Let 'f' be an extended real valued function with measurable domain D. Let $D_1 = \{x \in D : f(x) = \infty\}$ $D_2 = \{x : f(x) = -\infty\}$ Then 'f' is measurable if and only if D_1 and D_2 are measurable and the restriction of f to $D \setminus (D_1 \cup D_2)$ is a measurable function.

Proof:-

Let f be measurable on domain D.

Then,

$$D_1 = \{x \in D : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) > n\}$$

and

$$D_2 = \{x \in D : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) < -n\}$$

are measurable.

Consider the $D \setminus D_1 \cup D_2$ ... $\{x \in D : f(x) > \alpha\}$

$\{x \in D \setminus D_1 \cup D_2 : f(x) > \alpha\} = (D \setminus D_1 \cup D_2) \cap \{x \in D : f(x) > \alpha\}$

and the also restriction of f to $D_1 \cup D_2$ is measurable function ($\because$ $D \setminus D_1 \cup D_2$) is a measurable subset of D) $\because D = D_1 \cup D_2 \cup (D \setminus D_1 \cup D_2)$]

Conversely suppose that D_1, D_2 and the restriction of 'f' to $D_1 \cup D_2$ is measurable

Then

$\{x \in D : f(x) > \alpha\} = \{x \in D : f(x) = \infty\} \cup \{x \in (D \setminus D_1 \cup D_2) : f(x) > \alpha\}$

Being union of two measurables is measurable

And so 'f' is Measurable.

Theorem 1.11:-

If 'f' and 'g' are extended real valued measurable functions and α be a fixed number the $f+g$ is measurable provided we define $f+g$ to be α whenever it is of the form $\infty - \infty$ or $-\infty + \infty$

Proof:-

Define $D_1 = \{x \in D : f(x) = +\infty \text{ and } g(x) = -\infty\}$

$= \{x \in D : f(x) = +\infty\} \cap \{x \in D : g(x) = -\infty\}$

$D_2 = \{x \in D : f(x) = -\infty \text{ and } g(x) = +\infty\}$

$= \{x \in D : f(x) = -\infty\} \cap \{x \in D : g(x) = +\infty\}$

The sets D_1 and D_2 being intersection of two measurable sets are measurable.

Since $f+g$ is given to be a constant function on the domain $D_1 \cup D_2$ with value α, therefore $f+g$ is measurable on $D_1 \cup D_2$. Moreover $D_1 \cup D_2$ is measurable subset of D. So $f+g$ is measurable on $D \setminus (D_1 \cup D_2)$

In general if 'f' and 'g' are extended real valued measurable functions then the measurability of $f+g$ follows from the

equation and algebra of sets.

$$\{x : f(x) + g(x) > \alpha\} = \bigcup_{r \in Q} \Big[\{x : f(x) > r\} \cap \{x : g(x) > \alpha - r\}\Big]$$

$$\cup \Big[\{x : f(x) = \infty\} \setminus \{x : g(x) = -\infty\}\Big]$$

$$\cup \Big[\{x : g(x) = \infty\} \setminus \{x : f(x) = -\infty\}\Big]$$

Theorem 1.12 :-

Any extended real valued function defined on a ~~measurable~~ set of measure zero is measurable.

Proof

Let 'f' be an extended real valued function defined on a measurable set D with $m(D) = m^*(D) = 0$. We show that the set $D_1 = \{x \in D : f(x) > \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$

Since $D_1 \subseteq D$ and m^* is monotone

So $0 \le m^*(D_1) \le m^*(D) = 0$

$\Rightarrow m^*(D_1) = 0$

which implies that D_1 is a measurable set and so the result proved.

Definition 1.13

- A property is said to hold almost everywhere (a.e) if the set of points where this property fails to hold has measure zero.

- Two functions f and g with domain D are equal almost everywhere (a.e) symbolically written as $f = g$ a.e if and only if $m\{x : f(x) \neq g(x)\} = 0$

clearly if $f = g$ a.e. then $g = f$ a.e.

In pakistan water is a.e. b/c the places where [illegible] is not availables are countable so the set of those places has measure zero   313

Examples :-

1. Define $f : \mathbb{R} \to \{1, 2\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}' \\ 2 & \text{if } x \in \mathbb{Q} \end{cases}$$

Then

$f = 1$ a.e on $\mathbb{R}$ because $m(\mathbb{Q}) = 0$

Define

$$g : \mathbb{R} \to \mathbb{R} \text{ by } g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}' \\ x & \text{if } x \in \mathbb{Q} \end{cases}$$

then $f = g$ a.e.

Because $m\{x \in \mathbb{R} : f(x) \neq g(x)\} = m(\mathbb{Q} \setminus \{2\}) = 0$

b/c at $x = 2$
$f(x) = g(x)$

$(f = g \ \forall x \in \mathbb{Q}'$ and $f \neq g \ \forall x \in \mathbb{Q} \setminus \{2\})$

2. Define

$$f : \mathbb{R} \to \bar{\mathbb{R}} \text{ by } f(x) = \begin{cases} 5 & \text{if } x \in \mathbb{Q}' \\ \infty & \text{if } x \in \mathbb{Q} \end{cases}$$

Then 'f' is a.e finite because $m(\{x : f(x) = \infty\}) = 0$

Measure of every finite singleton and countably infinite set is zero.

- The sequence $\{f_n\}$ of functions defined on E is said to converge a.e (almost everywhere) to a function f if the set of points of $\{f_n\}$ does not converge to f has zero measure.

Theorem 1.15

Let 'f' and 'g' be extended real valued measurable functions which are finite almost everywhere. Then $f + g$ is measurable.

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Proof: Let 'f' and 'g' be extended real valued measurable functions which are finite almost everywhere.

Define

$D_1 = \{x \in D : f(x) = \infty \wedge g(x) = -\infty\}$

$= \{x \in D : f(x) = \infty\} \cap \{x \in D : g(x) = -\infty\}$

and $D_2 = \{x \in D : f(x) = -\infty$ and $g(x) = \infty\}$

$= \{x \in D : f(x) = -\infty\} \cap \{x \in D : g(x) = \infty\}$

then by the almost every where property

As f & g are finite a.e so D_1 and D_2 have measure zero ie $m(D_1) = 0$, $m(D_2) = 0$ and so $m(D_1 \cup D_2) = 0$.

Then $f+g$ is measurable on $D_1 \cup D_2$ and so $f+g$ is measurable on $D \setminus (D_1 \cup D_2)$.

° Any extended real valued func defined on set of measure zero is measurable

Theorem 1.16: Imp A/2012

Let 'f' be a measurable function with $f = g$ a.e on D. Then g is also measurable.

If f is measurable on $D \cup E$ then f is measurable on $D \setminus E$

Proof:

Put $E = \{x \in D : f(x) \neq g(x)\}$ Then by property almost everywhere the measure of E is zero. Further $D \setminus E$ is also measurable being compliment of a measurable set E.

Clearly for any $\alpha \in \mathbb{R}$,

$$\{x \in D : g(x) > \alpha\} = \{x \in D \setminus E : g(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\}$$

As $f = g$ on $D \setminus E$ so we get

$$\{x \in D : g(x) > \alpha\} = \{x \in D \setminus E : f(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\}$$

(i) $\{x \in D \setminus E : f(x) > \alpha\}$ is measurable as f is measurable on $D \setminus E$

(ii) The second set namely $\{x \in E : g(x) > \alpha\}$ is measurable as $\{x \in E : g(x) > \alpha\} \subseteq E$ and $m(E) = 0$

Hence the set $\{x \in D : g(x) > \alpha\}$ is measurable and so the result is proved. (*)

~~Theorem 1.1.8~~

~~Let $\{f_n\}$~~

Theorem 1.1.8 (Pre-discussion):

$f_i : [0, \pi/2] \to \mathbb{R}$

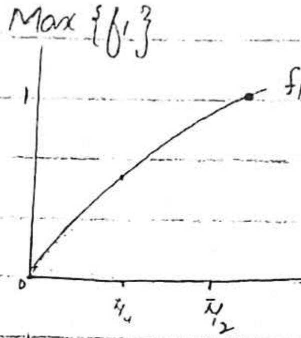
Let $f_1 = \sin x$

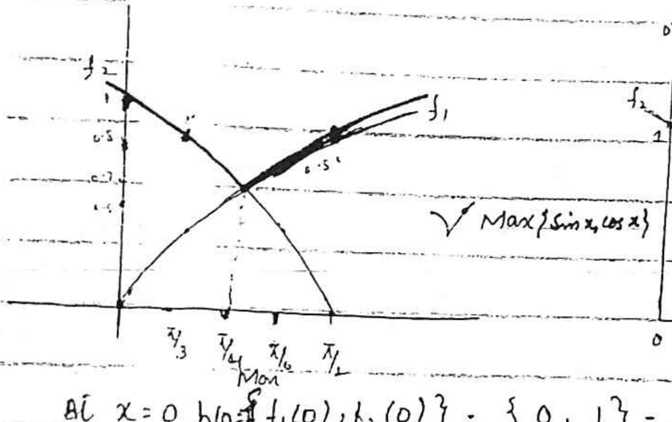
$f_2 = \cos x$

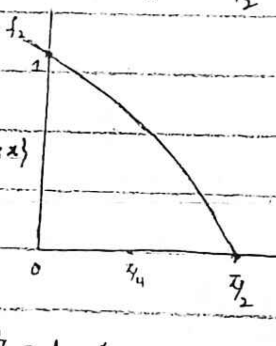
$\text{Max}\{f_1, f_2\}$

$h(x) = \text{Max}\{\sin x, \cos x\}$

First we draw graph for $\text{Max}\{f_1\}$







$\text{Max}\{f_2\}$

At $x = 0$ $h(0) = \text{Max}\{f_1(0), f_2(0)\} = \{0, 1\} = 1 = f_2$

At $x = \pi/6$ $h(\pi/6) = \text{Max}\{0.5, 0.866\} = 0.866 = f_2$

$x = \pi/4$ $h\pi/4 = \{0.707, 0.707\} = 0.707$

$x = \pi/3 = h(\pi/6) \cdot \{0.866, 0.5\} = f_1$

$x = \pi/2 = h\pi/2 = \{1, 0\} = f_1$

$g(x) = \text{Min}\{f_1, f_2\}$

$\text{Min}\{\sin x, \cos x\}$

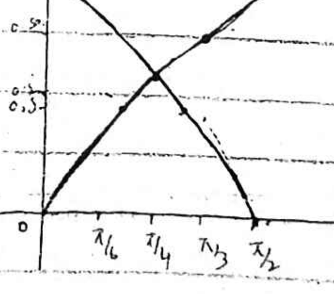
$x = 0$ $g(0) = \text{Min}\{0, 1\} = 0 = f_1$

$x = \pi/6$ $g(\pi/6) = \text{Min}\{0.5, 0.866\} = 0.5 = f_1$

$x = \pi/4$ $g(\pi/4) = \text{Min}\{0.707, 0.707\} = 0.707$

$x = \pi/3$ $g(\pi/3) = \text{min}\{0.866, 0.5\} = 0.5 = f_2$

$x = \pi/2$ $g(\pi/2) = \text{min}\{1, 0\} = 0$

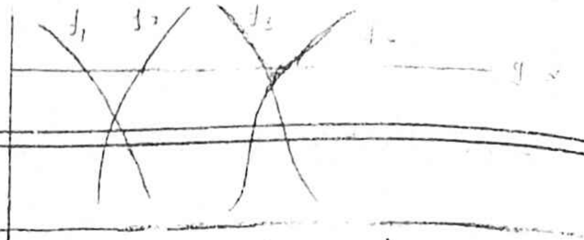


$\text{Min}\{\sin x, \cos x\}$

Theorem 1.18

Let $\{f_n\}$ be a sequence of extended real valued measurable function with same domain D then

(i) $\max_{1 \le i \le n} f_i$ is measurable for each n.

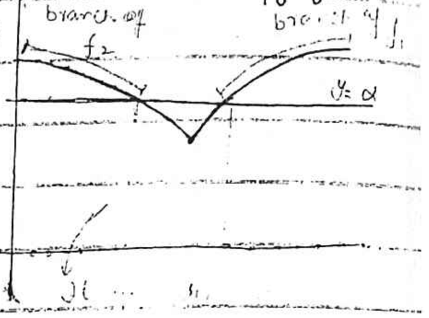


Let $\{f_n\}$ be sequence of extended real valued measurable function

Let $h = \max_{1 \le i \le n} f_i$

then for any $\alpha \in \mathbb{R}$, we prove that

$$\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$$



Let

$$y \in \{x : h(x) > \alpha\}$$

This gives that $h(y) > \alpha$

Now $h(x) = \max_{1 \le i \le n} f_i(x)$ so there exist some $j \in \{1, 2, 3, \ldots, n\}$ such that $h(x) = f_j(x)$. That is.

$$h(y) = f_j(y) > \alpha$$

This implies that $y \in \{x : f_j(x) > \alpha\}$ for some $j \in \{1, 2, 3, \ldots, n\}$.

Hence $y \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\} \Rightarrow \{x : h(x) > \alpha\} \subseteq \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$

Conversely,

Suppose that $y \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$

Then $y \in \{x : f_k(x) > \alpha\}$ for some $k \in \{1, 2, 3, \ldots, n\}$

i.e $f_k(y) > \alpha$. Also by definition

$$f_i(y) \le \max_{1 \le i \le n} f_i(y) = h(y) \ \forall \ i = 1, 2, 3, \ldots n$$

and so $f_k(y) \le h(y)$ which implies $h(y) > \alpha$.

Thus $y \in \{x : h(x) > \alpha\}$

$$\Rightarrow \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\} \subseteq \{x : h(x) > \alpha\}$$

Hence

$$\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

Finite union of measurable sets is measurable so $h = \max_{1 \le i \le n} f_i$ is measurable.

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(ii) $\min_{1 \le i \le n} f_i$ is measurable for each n

Let $\{f_n\}$ be the sequence of extended real valued measurable functions

Let $g = \min_{1 \le i \le n} f_i$

Then we show that for any $\alpha \in \mathbb{R}$

$\{x : g(x) > \alpha\} = \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$

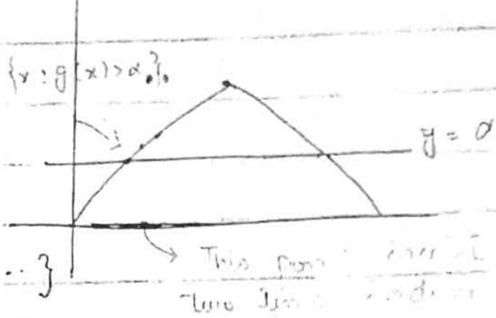
Let

$y \in \{x : g(x) > \alpha\}$

This gives $g(y) > \alpha$

Now $g(x) = \min_{1 \le i \le n} f_i(x)$

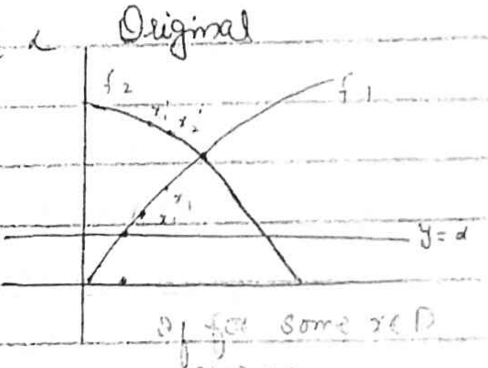
Then there exist $j \in \{1, 2, 3, \dots\}$



Such that

$g(y) = f_j(y) > \alpha$; for each i

Original



This implies that

$y \in \{x : f_j(x) > \alpha\}$

for each $j \in \{1, 2, 3, \dots\}$

Hence

$y \in \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$

If for some $x \in D$ $g(x) > \alpha$

$\{x : g(x) > \alpha\} \subseteq \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$

for x' $f_1(x) > \alpha$ also $f_2(x)$

the x point

two

Conversely suppose that

$y \in \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$

then

$y \in \{x : f_k(x) > \alpha\}$ for each k $\in \{1, 2, 3, \dots\}$

so we take intersection so that this

i.e. $f_k(y) > \alpha$ for each k

Then Definition

$g(y) = \min_{1 \le i \le n} f_i(y) = f_k$ ~~$> \alpha$ for some k~~

$\Rightarrow \min f_i(y) > \alpha \quad \forall i$

$\Rightarrow g(y) > \alpha$ for each α

$\Rightarrow y \in \{x : g(x) > \alpha\}$

$\Rightarrow \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\} \subseteq \{x : g(x) > \alpha\}$

Thus $\{x : g(x) > \alpha\} = \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$

being finite intersection of measurable sets is measurable so $g = \underset{1 \le i \le n}{\text{Min}} f_i$ is Measurable

(iii) $\underset{n \in N}{\text{Inf}} f_n$ is measurable.

Let $f = \underset{n \in N}{\text{Inf}} f_n$

Now we show that

$\{x : f(x) > \alpha\} = \bigcap_{i=1}^{\infty} \{x : f_i(x) > \alpha\}$

Let $y \in \{x : f(x) > \alpha\}$

then $f(y) > \alpha$

Obviously

$f(y) \le f_n(y)$ for each $n \in N$

So $\alpha < f(y) \le f_n(y) \quad \forall n$

$\Rightarrow y \in \{x : f_n(x) > \alpha\}$ for all $n \in N$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$

$\Rightarrow \{x : f(x) > \alpha\} \subseteq \bigcap_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$ — (i)

Conversely if

$y \in \bigcap_{n=1}^{\infty} \{x : f_n(x) > \alpha\} \quad \forall n \in N$

$\Rightarrow f_n(y) > \alpha \quad \forall n \in N$

that is $\alpha < f_n(y) \ \forall n \in N$

This gives 'α' is the lower bound of $\{f_1(y), f_2(y), \ldots\}$ by

$f(y) = \text{Inf}\{f_1(y), f_2(y), \ldots\}$.

A lower bound is always less than ~~or equal to~~ greatest lower bound so

$\alpha < f(y)$

$\Rightarrow y \in \{x : f(x) > \alpha\}$.

$\bigcap_{n=1}^{\infty} \{x : f_n(x) > \alpha\} \subseteq \{x : f(x) > \alpha\}$ — (ii)

$\{x : f(x) > \alpha\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$

The intersection of countable number of measurable sets is measurable. So 'f' must be Measurable.

$\sup_{n \in N} f_n$ is Measurable.

Let $h = \sup_{n \in N} f_n$

Now to show that h is Measurable we will show that $\{x : h(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$

Let $y \in \{x : h(x) > \alpha\}$

$\Rightarrow h(y) > \alpha$

$\because h(y) = \sup_{n \in N} f_n = f_m(y)$ for some $m \in N$

so $f_m(y) > \alpha$ for some $m \in N$

$\Rightarrow y \in \{x \in D : f_m(x) > \alpha\}$.

$\Rightarrow y \in \bigcup_{n=1}^{\infty} \{x \in D : f_n(x) > \alpha\}$

$\Rightarrow \{x : h(x) > \alpha\} \subseteq \bigcup_{n=1}^{\infty} \{x \in D : f_n(x) > \alpha\}$

Conversely suppose

$y \in \bigcup_{n=1}^{\infty} \{x \in D : f_n(x) > \alpha\}$

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$\Rightarrow y \in \{x \in D : f_m(x) > \alpha\}$ for some $m \in N$

$\Rightarrow f_m(y) > \alpha$ for some $m \in N$

By definition of supremum

$f_n(y) \leq \sup_{n \in N} f_n(y) \quad \forall n \in N$

In particular

$\alpha < f_m(y) \leq \sup_{n \in N} f_n(y)$

$\Rightarrow \sup_{n \in N} f_n(y) > \alpha$

$\Rightarrow h(y) > \alpha$

$\Rightarrow y \in \{x : h(x) > \alpha\}$

$\Rightarrow \bigcup_{n=1}^{\infty} \{x \in D : f_n(x) > \alpha\} \subseteq \{x : h(x) > \alpha\}$

Hence $\{x : h(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x \in D : f_n(x) > \alpha\}$

Being countable union of measurable sets is measurable & so

$h(x) = \sup_{n \in N} f_n(x)$

is measurable.

24-04-19

(v) $\overline{\lim} f_n = \lim \sup f_n$ is measurable

Let $\overline{\lim} f_n = \inf_n (\sup_{i \geq n} f_i) = \lambda$

Suppose that

$F_n = \sup \{ f_n, f_{n+1}, f_{n+2}, \dots f_{n+i}, \dots \}$ $(n = 1, 2, 3, \dots)$

Since each f_n is given to be measurable

So $\Rightarrow \sup_{i \geq n} f_n = F_n$ is measurable

Now $\{F_n\}$ is a sequence of measurable functions and $\lambda = \inf_n F_n$ so that λ being infimum of measurable functions is measurable.

(vi) $\underline{\lim} f_n = \lim \inf f_n$ is measurable

Let

$\underline{\lim} f_n = \sup_n (\inf_{i \geq n} f_i) = u$

Suppose that

$G_n = \inf \{ f_n, f_{n+1}, f_{n+2}, \dots, f_{n+i}, \dots \}$ $(n = 1, 2, 3, \dots)$

Since each f_n is given to be measurable

So $G_n = \inf_{i \geq n} f_n$ is measurable

Now $\{G_n\}$ is a sequence of measurable functions and $u = \sup_n G_n$ so 'u' being supremum of sequence of measurable functions is measurable.

(vii) If $\lim_{n \to \infty} f_n(x) = f(x)$ exists then 'f' is measurable

Given that $\lim_{n \to \infty} f_n(x) = f(x) = \overline{\lim} f_n = \underline{\lim} f_n$

As both $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are measurable by (v) and (vi) so 'f' should be measurable.

v.v. Imp — Imp A/2014

Theorem 1.19 :- A continous function defined on a measurable set is measurable.

Proof :-

Let $f: D \to R$ be a continous function defined on a measurable domain $D \subseteq R$

For measurability of 'f', it is sufficient to prove that $\{x : f(x) > \alpha\}$ is a measurable set for each $\alpha \in \overline{R}$

$\{x \in D : f(x) > \alpha\} = \{x \in D : f(x) \in (\alpha, \infty)\} = f^{-1}(\alpha, \infty)$

Since $f: D \to R$ is continous and (α, ∞) is open in R then $f^{-1}(\alpha, \infty)$ is an open subset in D. (Here D is a subspace of R with relative topology on D)

$f(x) \in (\alpha, \infty)$
$x \in f^{-1}(\alpha, \infty)$

Then by definition of Relative topology on D, there exist an open set F in R such that
$f^{-1}(\alpha, \infty) = D \cap F$

$Y \subseteq X$
$U = V \cap Y$
(open in Y) (open in X)

The set F being open set is measurable so
$\Rightarrow f^{-1}(\alpha, \infty)$ is Measurable

D = Measurable
F is open in R so is measurable.
Thus $D \cap F$ is Measurable

$\Rightarrow \{x \in D : f(x) > \alpha\}$ is Measurable

Hence a continous function defined

on a measurable set is measurable. (X)

THEOREM 1.20

Let 'f' be a measurable function and G be an open set. Then $\{x : f(x) \in G\}$ is a measurable set.

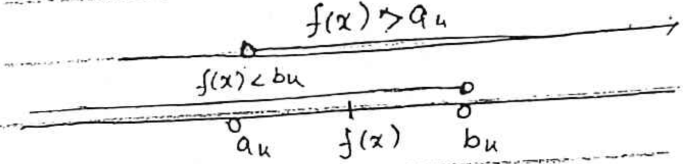
Proof:

As we know that

"~~For every~~ Any non empty open set G in $\mathbb{R}$ is the union of a countable collection of disjoint open intervals.

i.e $G = \bigcup_{k=1}^{\infty} I_k$ where $I_k = (a_k, b_k)$ are pairwise disjoint open intervals.

$\{x : f(x) \in G\} = \{x : f(x) \in \bigcup_{k=1}^{\infty} (a_k, b_u)\}$



$= \bigcup_{k=1}^{\infty} \{x : f(x) \in (a_k, b_u)\}$

$\{x : f(x) \in G\} = \bigcup_{k=1}^{\infty} [\{x : f(x) > a_u\} \cap \{x : f(x) < b_u\}]$

Since 'f' is measurable & so $\{x : f(x) > a_u\}$ and $\{x : f(x) < b_u\}$ are measurable and so their intersection

Hence $\{x : f(x) \in G\}$ being countable union of Measurable sets is measurable. (X)

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Theorem 1.21

Let 'f' and 'g' be two measurable functions defined on the same set D. Then the following sets are measurable.

(i) $\{x : f(x) > g(x)\}$

Proof:- We need

To show that $\{x : f(x) > g(x)\}$ is measurable.

As we know that $f(x), g(x) \in \mathbb{R}$

and Between any two real number there exists a rational number such that

for $f(x) > g(x)$

$\exists$ r s.t

$f(x) > r > g(x)$

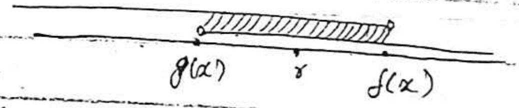
To show that $\{x : f(x) > g(x)\}$ is measurable

we prove that

$\{x : f(x) > g(x)\} = \bigcup_{r \in Q} [\{x : f(x) > r\} \cap \{x : g(x) < r\}]$

Let $y \in \{x : f(x) > g(x)\}$

then $f(y) > g(y)$



Then by the above hypothesis $\exists$ a rational no. 'r'

s.t $f(y) > r > g(y)$

$f(y) > r$ and $g(y) < r$

$\Rightarrow y \in \{x : f(x) > r\}$ & $y \in \{x : g(x) < r\}$

$\Rightarrow y \in \{x : f(x) > r\} \cap \{x : g(x) < r\}$

$\Rightarrow y \in \bigcup_{r \in Q} [\{x : f(x) > r\} \cap \{x : g(x) < r\}]$

$\{x : f(x) > g(x)\} \subseteq \bigcup_{r \in Q} [\{x : f(x) > r\} \cap \{x : g(x) < r\}]$

Conversely,

Suppose that

$y \in \bigcup_{r \in Q} [\{x : f(x) > r\} \cap \{x : g(x) < r\}]$

$\Rightarrow y \in \{x : f(x) > r\} \cap \{x : g(x) < r\}$

$\Rightarrow f(y) > r$ and $r > g(y)$

Thus we get

$f(y) > r > g(y)$ $\Rightarrow f(y) > g(y)$

$\Rightarrow y \in \{f(x) > g(x)\}$

$\Rightarrow \bigcup_{r \in Q} [\{x : f(x) > r\} \cap \{x : g(x) < r\}] \subseteq \{x : f(x) > g(x)\}$

Thus $\{x : f(x) > g(x)\} = \bigcup_{r \in Q} [\{x : f(x) > r\} \cap \{x : g(x) < r\}]$

Since 'f' and 'g' are Measurable functions

So $\{x : f(x) > r\}$

and $\{x : g(x) < r\}$

S.O R.H.S of above relation is measurable

being union of Measurable sets

So $\{x : f(x) > g(x)\}$ is Measurable

(ii) $\{x : f(x) < g(x)\}$

To show that $\{x : f(x) < g(x)\}$ is measurable

we will show that $\{x : f(x) < g(x)\}$

$= \bigcup_{r \in Q} [\{x : f(x) < r\} \cap \{x : g(x) > r\}]$

Let $y \in \{x : f(x) < g(x)\}$ then

$f(y) < g(y)$

Since $f(y)$ and $g(y) \in \mathbb{R}$

So there exist a rational no. 'r' such that

$f(y) < r < g(y)$

$\Rightarrow f(y) < r$ and $g(y) > r$

$\Rightarrow y \in \{x : f(x) < r\}$ and $y \in \{x : g(x) > r\}$

$\Rightarrow y \in \{x : f(x) < r\} \cap \{x : g(x) > r\}$

$y \in \bigcup_{r \in Q} [\{x : f(x) < r\} \cap \{x : g(x) > r\}]$

$\Rightarrow \{x : f(x) < g(x)\} \subseteq \bigcup_{r \in Q} [\{x : f(x) < r\} \cap \{x : g(x) > r\}]$

Conversely suppose

that $y \in \bigcup_{r \in Q} [\{x : f(x) < r\} \cap \{x : g(x) > r\}]$

$\Rightarrow y \in \{x : f(x) < r\} \cap \{x : g(x) > r\}$

$\Rightarrow y \in \{x : f(x) < r\}$ and $y \in \{x : g(x) > r\}$

$\Rightarrow f(y) < r$ and $g(y) > r$

$\Rightarrow f(y) < r < g(y)$

$\Rightarrow f(y) < g(y)$

$\Rightarrow y \in \{x : f(x) < g(x)\}$

$\Rightarrow \bigcup_{r \in Q} [\{x : f(x) < r\} \cap \{x : g(x) > r\}] \subseteq \{x : f(x) < g(x)\}$

Hence $\{x : f(x) < g(x)\} = \bigcup_{r \in Q} [\{x : f(x) < r\} \cap \{x : g(x) > r\}]$

Since R.H.S being countable union of two measurable sets is measurable

so $\{x : f(x) < g(x)\}$ is Measurable

(iii) $\{x : f(x) \le g(x)\}$

As

$D = \{x : f(x) > g(x)\} \cup \{x : f(x) \le g(x)\}$

We have shown above that

$\{x : f(x) > g(x)\}$ is measurable subset of domain D.

So $D \setminus \{x : f(x) > g(x)\} = \{x : f(x) \le g(x)\}$ is measurable.

(iv) $\{x : f(x) = g(x)\}$

To show that $\{x : f(x) = g(x)\}$

we will show that

$\{x : f(x) = g(x)\} = \{x : f(x) \le g(x)\} \setminus \{x : f(x) < g(x)\}$

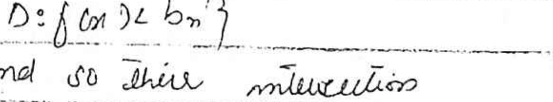
As $\{x : f(x) \leq g(x)\}$ and $\{x : f(x) < g(x)\}$
both are Measurable sets.
So R.H.S is measurable
And hence $\{x : f(x) = g(x)\}$ is Measurable (Being difference of two measurable sets)
*

Theorem 1.22:- Imp A/2013/2011
Let 'f' be any real valued function defined on a measurable domain D and G an open set in $\mathbb{R}$. Then 'f' is measurable if and only if $f^{-1}(G)$ is Measurable.

Proof:-
Let 'f' be measurable and G be an open set in $\mathbb{R}$.

As every open set is countable union of open intervals so
$G = \bigcup_{n=1}^{\infty} I_n$ where $I_n = (a_n, b_n)$
are disjoint open intervals.
Since $I_n = (a_n, b_n) = (-\infty, b_n) \cap (a_n, \infty)$
So

$f^{-1}(I_n) = f^{-1}[(a_n, b_n)]$



$= f^{-1}[(-\infty, b_n) \cap (a_n, \infty)]$

$f^{-1}(I_n) = \{x \in D : f(x) < b_n\} \cap \{x \in D : f(x) > a_n\}$
Since f is measurable so
$\{x \in D : f(x) > a_n\}$ & $\{x \in D : f(x) < b_n\}$
are measurable. and so their intersection
Hence $f^{-1}(I_n) = f^{-1}[(-\infty, b_n)] \cap f^{-1}[(a_n, \infty)]$
is Measurable. for each n
~~Thus~~ and So

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$f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n)$ is measurable.

Conversely suppose that $f^{-1}(G)$ is a measurable set for any open set G in R. In particular for

$G = (\alpha, \infty)$ $\alpha \in \mathbb{R}$

$f^{-1}(G) = f^{-1}(\alpha, \infty) = \{x : f(x) > \alpha\}$

is a measurable set

Hence 'f' is measurable.

by continuous function theorem

<u>For Extended Real Valued function f:</u>

The above Result holds for the extended real valued function f

Suppose 'f' is measurable

Define $\Omega = \{E \subseteq \bar{R} : f^{-1}(E) \in M\}$

which is a σ-algebra on $\bar{R}$.

Now $f^{-1}(\alpha_n, \infty] \in M$ for each n

as $f^{-1}(\alpha_n, \infty] \in M$ for each $\alpha \in \bar{\mathbb{R}}$ and

$\alpha_n \to \alpha$ as $n \to \infty$

$[-\infty, \alpha) = \bigcup_{n=1}^{\infty} [-\infty, \alpha_n] = \bigcup_{n=1}^{\infty} (\alpha_n, \infty]'$ and so on

$[-\infty, \alpha) \in \Omega$ Hence

$(\alpha, \beta) = [-\infty, \beta) \cap (\alpha, \infty] \in \Omega$

As every open set G in R is countable union of open intervals so $G \in \Omega$ which gives $f^{-1}(G)$ is measurable.

Conversely suppose that $f^{-1}(G)$ is measurable for any open set G in $\bar{R}$

To prove $f^{-1}(G)$ is meas
$\Omega = \{E \subseteq \bar{R} : f^{-1}(E) \in M\}$
Measurable ⇒ Lebesgue measure

$f^{-1}(\alpha, \infty] = \{x : f(x) > \alpha\}$ is Measurable $\forall \alpha \in R$
$f(x) \in (\alpha, \infty]$

As $\bigcup_{n=1}^{\infty} [a + 1/n, b - 1/n] = (a, b)$

for each n
$f^{-1}(\alpha_n, \infty] \in M$ for each n
$\Rightarrow (\alpha_n, \infty] \in \Omega$
$(\alpha_n, \infty]' = [-\infty, \alpha_n] \in \Omega$

* $G = (\alpha, \infty]$ open in $\bar{R}$ $[-\infty, \infty]$ 329

In particular, if $G = (\alpha, \infty]$ $\alpha \in \bar{R}$ then

$f^{-1}(G) = f^{-1}(\alpha, \infty] = \{x : f(x) > \alpha\}$ is

a measurable set.

Hence 'f' is Measurable function.

Corollary 1.23

If F is closed subset of $\bar{R}$ then the set $f^{-1}(F)$ is measurable if and only if f is measurable.

~~Result~~.

Suppose that 'f' is measurable then

$F' = G$ is open.

As $f^{-1}(G) = f^{-1}(F') = [f^{-1}(F)]'$ is measurable

by above result so

$f^{-1}(F)$ is measurable.

Conversely suppose that $f^{-1}(F)$ is measurable for any closed subset F in $\bar{R}$. Let G be any open set in $\bar{R}$. As G' is closed so by hypothesis $f^{-1}(G') = [f^{-1}(G)]'$ is measurable

and hence $f^{-1}(G)$ is measurable

Then f is Measurable. (X)

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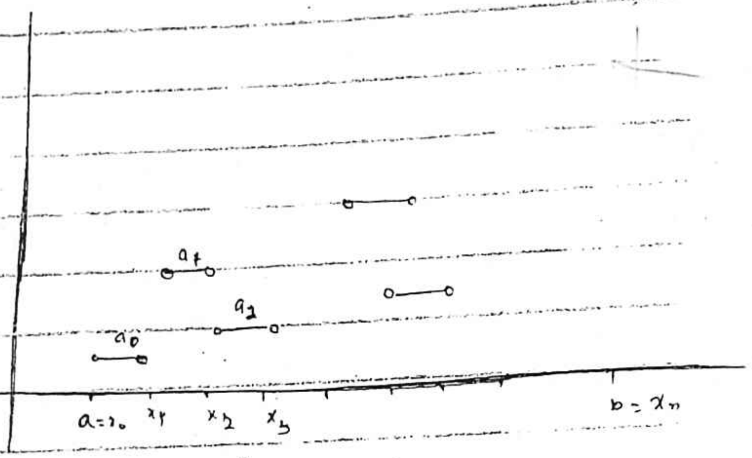
A|2013 A|2012

~~Simple~~ Step Function:

Let 'ϕ' be a real valued function defined on $[a,b]$. We say 'ϕ' is a step function if there is a partition of $[a,b]$, $a = x_0 < x_1 < x_2, \ldots < x_i < x_{i+1} < \ldots < x_n = b$ such that ϕ assumes only one value in the intervals (x_{i-1}, x_i), $i = 1, 2, 3, \ldots, n$. That is a step function is a constant function in the interval (x_{i-1}, x_i) $\forall\ i = 1, 2, 3, \ldots, n$

$\phi(x) = a_0 \quad x \in (x_0, x_1)$
$a_1 \quad (x \in (x_1, x_2)$
$\vdots$
$a_{n-1}, x \in (x_{n-1}, x_n)$



• Steps are mixed not in increasing order

Step function

Theorem 2.2 A|2013 A|2012

~~Inf~~ A step function 'f' with measurable domain is measurable

Let $\alpha \in \mathbb{R}$ and 'f' assumes finite number of values $a_1, a_2, a_3, \ldots, a_n$ on domain $[a,b]$.

Case i: Let $\alpha < \min\{a_1, a_2, a_3, \ldots a_n\}$

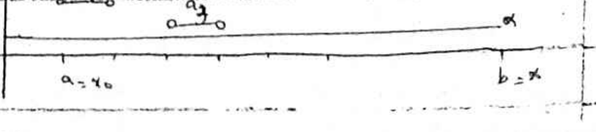
Then

$\{x : f(x) > \alpha\} = [a,b]$

($\because$ every closed interval is Measurable)

So $\{x : f(x) > \alpha\}$ is Measurable

and so 'f' is Measurable.



Case ii :- If $\alpha \geq Max\{a_1, a_2, a_3, \ldots, a_n\}$

Then

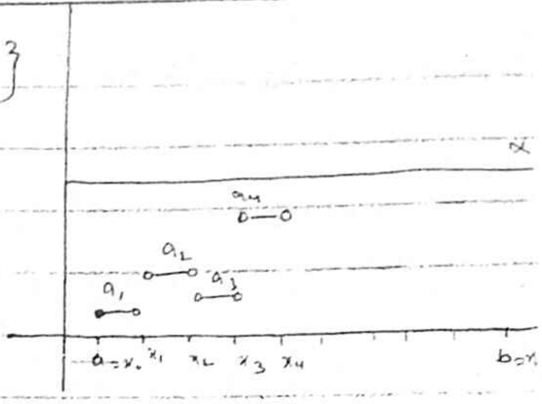
we get

$\{x : f(x) > \alpha\} = \phi$

(Since ϕ is measurable)

So $\{x : f(x) > \alpha\}$ is Measurable

And hence 'f' is measurable.



Case iii :-

If $min\{a_1, a_2, a_3, \ldots, a_n\} \leq \alpha < Max\{a_1, a_2, a_3, \ldots, a_n\}$

then $\{x : f(x) > \alpha\}$ is the union.

of intervals (x_i, x_{i+1})

Such that $f(x) = a_j$ in

(x_j, x_{j+1}) and

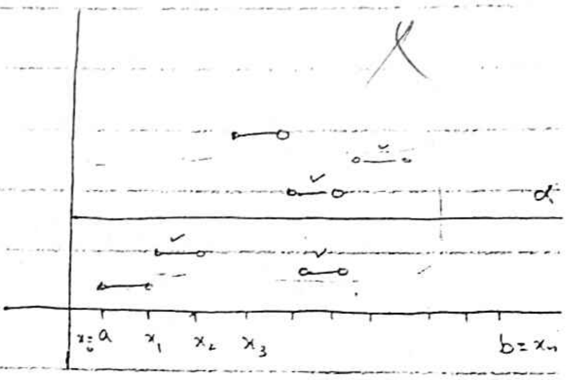
$\alpha < a_j \leq max\{a_1, a_2, a_3 \ldots a_n\}$

then (x_j, x_{j+1}) is measurable

and so $\{x : f(x) > \alpha\}$ being

the finite union of measurable

sets is measurable



∵ every open interval is measurable

Definition 2·3:-

Let A be any subset of a nonempty set X. Then the characteristic function of the set A denoted by X_A is the real valued function defined by

$$X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \ (\text{ie } x \in A') \end{cases}$$

Theorem 2·4

The function X_A defined on a measurable set D is measurable if and only if A is a measurable subset of D.

Proof :-

Let X_A be measurable function on D. Then

D being domain of a measurable function is measurable. Moreover, the set $\{x \in D : \chi_A(x) > \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$

In particular for $\alpha = 0$

The set $\{x \in D : \chi_A(x) > 0\} = \{x \in D : \chi_A(x) = 1\} = A$ must be measurable subset of D as required.

Now conversely, suppose that A is measurable. By hypothesis D is measurable, set

$\phi = A \cap D \setminus A$ is measurable.

Now for any

$$\alpha \in \mathbb{R} \Rightarrow \{x \in D : \chi_A(x) > \alpha\} = \begin{cases} A & \text{if } 0 \leq \alpha < 1 \\ \phi & \text{if } \alpha \geq 1 \\ D & \text{if } \alpha < 0 \end{cases}$$

$\{x \in D : \chi_A(x) > \alpha\}$ for $\alpha < 0$

$\chi_A(x) = 0$ or $\chi_A(x) = 1$

$\Rightarrow x \in A$ or $x \in A'$

$\Rightarrow x \in A \cup A' = D$.

Hence χ_A is a Measurable function.

Corollary: 2.5

The Dirichlet's function 'f' defined on $[0,1]$ is a measurable function.

Proof:- Dirichlet's function is defined as

$$f(x) = \chi_E = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \text{ ie } x \in Q' \end{cases}$$

Let L be a measurable subset of $[0,1]$

As every closed interval is measurable so $[0,1]$ is measurable. Also $\phi = [0,1] \setminus L$ is measurable

Case ii :- If $\alpha \geq Max\{a_1, a_2, a_3, \dots, a_n\}$

Then

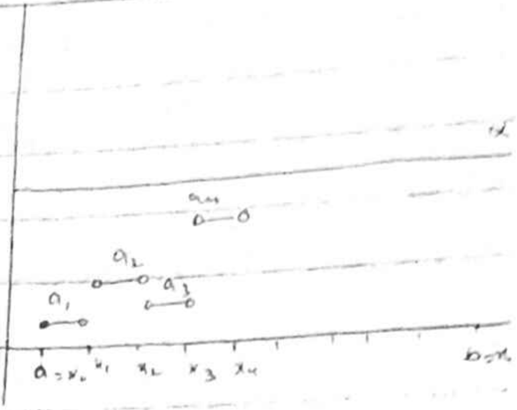
we get

$\{x : f(x) > \alpha\} = \phi$

(Since ϕ is measurable)

So $\{x : f(x) > \alpha\}$ is Measurable

And hence f is measurable



Case iii :-

If $min\{a_1, a_2, a_3, \dots, a_n\} \leq \alpha < Max\{a_1, a_2, a_3, \dots, a_n\}$

then $\{x : f(x) > \alpha\}$ is the union.

of intervals (x_i, x_{i+1})

Such that $f(x) = a_j$ in

(x_j, x_{j+1}) and

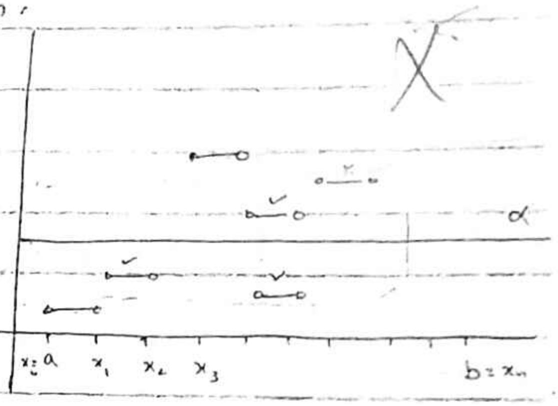
$\alpha < a_j \leq max\{a_1, a_2, a_3 \dots a_n\}$

then (x_j, x_{j+1}) is measurable

and so $\{x : f(x) > \alpha\}$ being

the finite union of measurable

sets is measurable



$\because$ every open interval is measurable

Definition 2.3 :-

Let A be any subset of a nonempty set X. Then the characteristic function of the set A denoted by χ_A, is the real valued function defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \text{ (ie } x \in A') \end{cases}$$

Theorem 2.4

The function χ_A defined on a measurable set D is measurable if and only if A is a measurable subset of D.

Proof :-

Let χ_A be measurable function on D Then

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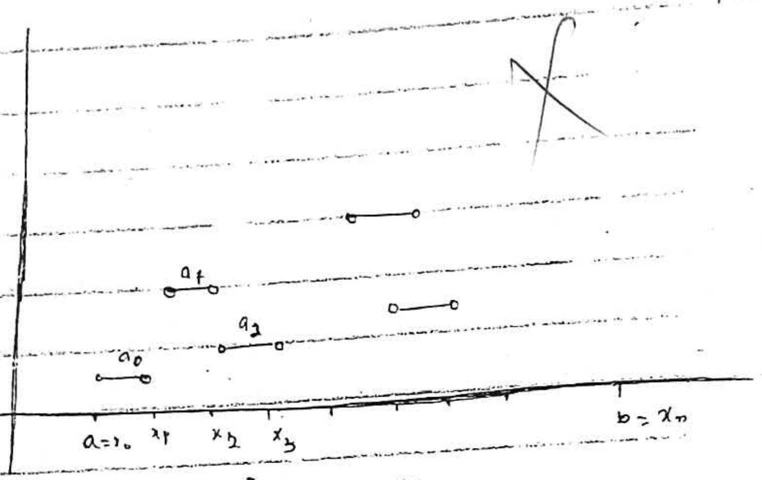
~~Simple~~ Step Function:

Let ϕ' be a real valued function defined on $[a,b]$. We say ϕ' is a step function if there is a partition of $[a,b]$, $a = x_0 < x_1 < x_2, \dots < x_i < x_{i+1} < \dots < x_n = b$ such that ϕ assumes only one value in the intervals (x_{i-1}, x_i), $i = 1,2,3,\dots,n$. That is a step function is a constant function in the interval (x_{i-1}, x_i) $\forall$ $i = 1,2,3,\dots,n$

$\phi(x) = a_0$, $x \in (x_0, x_1)$
a_1 $(x \in (x_1, x_2))$
$\vdots$
a_{n-1}, $x \in (x_{n-1}, x_n)$

- Steps are mixed not in increasing order.



Step function.

Theorem 2.2 A/2013 A/2012

A step function 'f' with measurable domain is measurable.

Let $\alpha \in \mathbb{R}$ and 'f' assumes finite number of values $a_1, a_2, a_3, \dots, a_n$ on domain $[a,b]$.

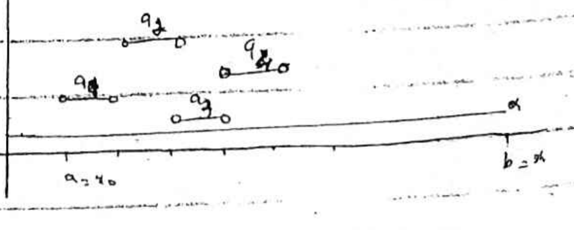
Case i: Let $\alpha < \min\{a_1, a_2, a_3, \dots, a_n\}$

Then

$\{x : f(x) > \alpha\} = [a,b]$

($\because$ every closed interval is Measurable)

so $\{x : f(x) > \alpha\}$ is Measurable and so 'f' is Measurable.



* $G = (\alpha, \infty]$ open in $\bar{R}$ $[-\infty, \infty]$

In particular, if $G = (\alpha, \infty]$ $\alpha \in \bar{R}$ then
$f^{-1}(G) = f^{-1}(\alpha, \infty] = \{x : f(x) > \alpha\}$ is
an measurable set.
Hence 'f' is Measurable function.

Corollary 1.23

If F is closed subset of $\bar{R}$ then the set $f^{-1}(F)$ is measurable if and only if f is measurable ~~result~~.

Suppose that 'f' is measurable then
$F' = G$ is open.
As $f^{-1}(G) = f^{-1}(F') = [f^{-1}(F)]'$ is measurable
by above result so
$f^{-1}(F)$ is measurable.
Conversely suppose that $f^{-1}(F)$ is measurable for any closed subset F in $\bar{R}$. Let G be any open set in $\bar{R}$. As G' is closed so by hypothesis $f^{-1}(G') = [f^{-1}(G)]'$ is measurable
and hence $f^{-1}(G)$ is measurable
Then f is Measurable.

ڈیسنٹ فوٹو کاپی سنٹر

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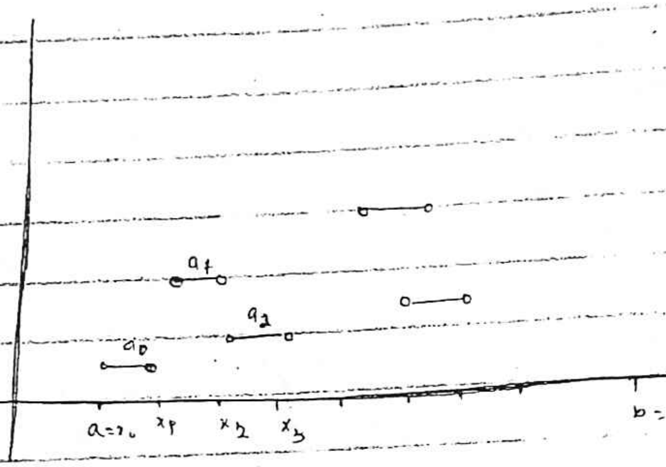
~~Simple~~ Step FUNCTION:

Let 'φ' be a real valued function defined on [a, b]. We say 'φ' is a step function if there is a partition of [a, b], $a = x_0 < x_1 < x_2, \ldots < x_i < x_{i+1} < \ldots < x_n = b$ such that φ assumes only one value in the intervals (x_{i-1}, x_i), $i = 1, 2, 3, \ldots, n$. That is a step function is a constant function in the interval (x_{i-1}, x_i) $\forall\ i = 1, 2, 3, \ldots, n$

$\phi(x) = a_0 \quad x \in (x_0, x_1)$
$a_1 \quad (x \in (x_1, x_2)$
$\vdots$
$a_{n-1}, x \in (x_{n-1}, x_n)$



- Steps are mixed not in necessary order.

Step function

Theorem 2.2 A/2013 A/2012

~~Inf~~ A step function 'f' with measurable domain is measurable

Let $\alpha \in \mathbb{R}$ and 'f' assumes finite number of values $a_1, a_2, a_3, \ldots, a_n$ on domain [a, b].

Case i: Let $\alpha < \min\{a_1, a_2, a_3, \ldots a_n\}$

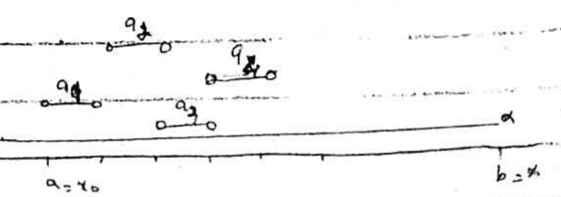
Then

$\{x : f(x) > \alpha\} = [a, b]$

(∵ every closed interval is Measurable)

So $\{x : f(x) > \alpha\}$ is Measurable

and so 'f' is Measurable.



Case ii: If $\alpha \geq Max\{a_1, a_2, a_3, \ldots, a_n\}$

Then

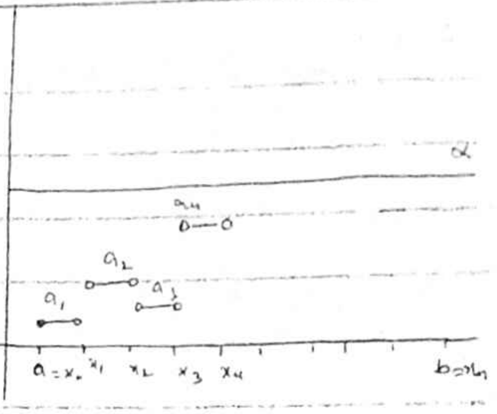
we get

$\{x : f(x) > \alpha\} = \phi$

(Since ϕ is measurable)

So $\{x : f(x) > \alpha\}$ is Measurable

And hence 'f' is measurable.



Case iii:

If $min\{a_1, a_2, a_3, \ldots, a_n\} \leq \alpha < Max\{a_1, a_2, a_3, \ldots, a_n\}$

then $\{x : f(x) > \alpha\}$ is the union of intervals (x_i, x_{i+1})

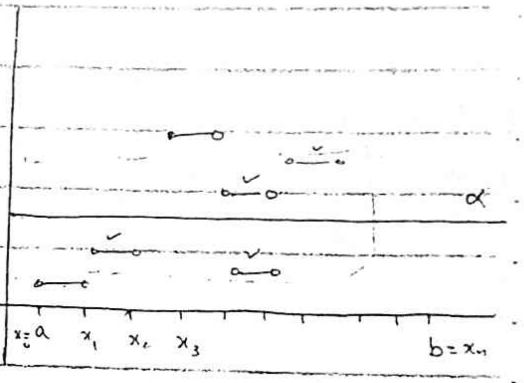
such that $f(x) = a_j$ in (x_j, x_{j+1}) and

$\alpha < a_j \leq max\{a_1, a_2, a_3 \ldots a_n\}$

then (x_j, x_{j+1}) is measurable

and so $\{x : f(x) > \alpha\}$ being the finite union of measurable sets is measurable

$\because$ every open interval is measurable



Definition 2·3: Characteristics function;

Let A be any subset of a nonempty set X. Then the characteristic function of the set A denoted by X_A, is the real valued function defined by

$$X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \ (\text{ie } x \in A') \end{cases}$$

Theorem 2·4

The function X_A defined on a measurable set D is measurable if and only if A is a measurable subset of D.

Proof:-

Let X_A be measurable function on D. Then

D being domain of a measurable function is measurable. Moreover, the set $\{x \in D : \chi_A(x) > \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$

In particular for $\alpha = 0$

The set $\{x \in D : \chi_A(x) > 0\} = \{x \in D : \chi_A(x) = 1\} = A$ must be measurable subset of D as required

Now conversely, suppose that A is measurable.

By hypothesis D is measurable set

$\therefore$ $\phi = A \cap D \setminus A$ is measurable

ϕ is measurable;

Now for any

$$\alpha \in \mathbb{R} = \{x \in D : \chi_A(x) > \alpha\} = \begin{cases} A & \text{if } 0 \le \alpha < 1 \\ \phi & \text{if } \alpha \geq 1 \\ D & \text{if } \alpha < 0 \end{cases}$$

$\{x \in D : \chi_A(x) > \alpha\}$ for $\alpha < 0$

$\chi_A(x) = 0$ or $\chi_A(x) 1$

$\Rightarrow x \in A$ or $x \in A'$

$\Rightarrow x \in A \cup A' = D$

Hence χ_A is a Measurable function.

Corollary: 2·5

The Dirichlet's function 'f' defined on $[0,1]$ is a measurable function.

Proof:- Dirichlet's function is defined as

$$f(x) = \chi_L = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \text{ ie } x \in Q' \end{cases}$$

Let L be a measurable subset of $[0,1]$

As every closed interval is measurable so $[0,1]$ is measurable. Also $\phi = [0,1] \cap [0,1] \setminus L$ is measurable

Not for any $\alpha \in \mathbb{R}$

$$\alpha \in \mathbb{R},\ \{x \in D : \chi_A(x) > \alpha\} = \begin{cases} A & \text{if } 0 < \alpha < 1 \\ \phi & \text{if } \alpha \geq 1 \\ [0,1] & \text{if } \alpha < 0 \end{cases}$$

Hence $f(x) = \chi_L$ is a measurable function

Remarks :-

From above two theorems, for A set A is non measurable if and only if characteristic χ_A is non measurable.

Definition 2.6 :- "Simple function"

A real valued function defined on $A \subseteq X$ is said to be a simple function if it assumes only a finite number of values say $a_1, a_2, a_3, \ldots, a_n$.

That is range of ϕ is a finite set

$\therefore$ Range of characteristic function

$\chi_A = [0,1]$.

Lemma Theorem 2.7#

If ϕ is a simple function defined on A and attains distinct values $a_1, a_2, a_3, \ldots, a_n$ then $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$, where $A_i = \{x \in A : \phi(x) = a_i\}$ with $\bigcup_{i=1}^{n} A_i = A$.

If $x \in A$ then $\phi(x) = a_i$ then by definition of A_i $x \in A_i$ and $x \notin A_i'$

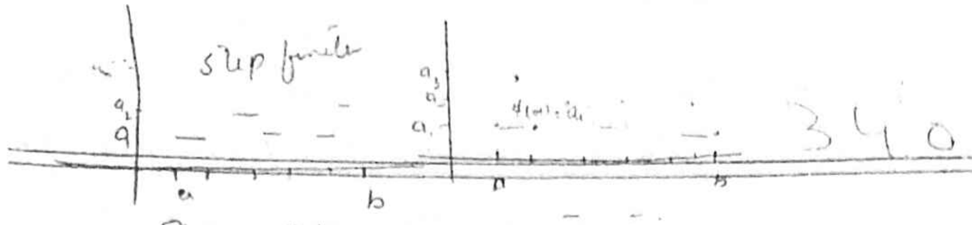
$$\text{Hence } \chi_{A_i}(x) = \begin{cases} 1 & x \in A_i \\ 0 & x \in A_i' \text{ ie } x \in A_j \quad j \neq i \end{cases}$$

$A_i \cap A_j = \phi$

$$\phi(x) = a_i = a_i(1)$$

$$= a_1 \chi_{A_1}(x) + a_2 \chi_{A_2}(x) + \ldots + a_i \chi_{A_i}(x) + \ldots + a_n \chi_{A_n}(x)$$

$$= \sum_{i=1}^{n} a_i \chi_{A_i}(x) \qquad \because \chi_{A_j}(x) = 0 \text{ if } j \neq i$$



Then above representation of Simple function ϕ is called canonical or standard representation

Example 2.9

Show that if $A, B \subseteq X$, then

(i) $\chi_{A\cap B} = \chi_A \cdot \chi_B$

Proof:- Clearly the domain of functions on both sides of any equality X

Case I :- If $x \in A\cap B$, then $x \in A$ and $x \in B$

So $\chi_{A\cap B}(x) = 1 = \chi_A(x)\chi_B(x)$

$\chi_{A\cap B} = \chi_A \cdot \chi_B$

Case II :- If $x \notin A\cap B$ then

$\chi_{A\cap B}(x) = 0$

In case $x \notin A$ $\chi_A(x) = 0$ and $x \in B$ $\chi_B(x) = 1$

$\chi_{A\cap B} = 0 = \chi_A \cdot \chi_B$

In case $x \in A$ and $x \notin B$ Then

$\chi_A(x) = 1$ and $\chi_B(x) = 0$

In case $x \notin A$ and $x \notin B$

Then $\chi_A(x) = 0$ and $\chi_B(x) = 0$

In all the cases

$\chi_{A\cap B}(x) = \chi_A(x) \cdot \chi_B(x)$

(ii) $\chi_{A\cup B}(x) = \chi_A + \chi_B - \chi_A \cdot \chi_B$

Suppose $x \in A\cup B$ Then there are three possibilities

(a) $x \in A$ and $x \in B$ (b) $x \notin A$ and $x \in B$

(c) $x \in A$ and $x \notin B$

In case (a), (b) & (c)

(a) $\Rightarrow$ $\chi_{A\cup B}(x) = 1 = \chi_A(x) + \chi_B(x) - \chi_A \cdot \chi_B(x)$

$= 1 + 1 - 1$

$\Rightarrow \chi_{A\cup B} = 1 = \chi_A + \chi_B - \chi_A \cdot \chi_B$

(b) $\chi_{A\cup B}(x) = 1 = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$

$= 0 + 1 - 0$

$\Rightarrow \chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$

(c) $\Rightarrow \chi_{A\cup B}(x) = 1 = \chi_A(x) + \chi_B(x) - \chi_A \cdot \chi_B$

$= 1 + 0 + 0$

$\Rightarrow \chi_{A\cup B} = 1 = \chi_A + \chi_B - \chi_A \cdot \chi_B$

Suppose $x \notin A\cup B$ Then $x \notin A$ & $x \notin B$

$\chi_{A\cup B}(x) = 0 = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$

$\Rightarrow \chi_{A\cup B} = \chi_A + \chi_B - \chi_A\chi_B$

(iii) $\chi_{A\cup B} = \chi_A + \chi_B$ if $A\cap B = \phi$

From above (ii)

If $A\cap B = \phi$

Then $x \notin A$ and $x \in B$ or $x \in A$ and $x \notin B$

$\Rightarrow \chi_{A\cup B}(x) = 1 = \chi_A(x) + \chi_B(x)$

$= 0 + 1$

$\Rightarrow \chi_{A\cup B} = \chi_A + \chi_B$

If $x \in A$ and $x \notin B$

$\chi_{A\cup B}(x) = 1 = \chi_A(x) + \chi_B(x)$

$= 1 + 0$

$\chi_{A\cup B} = \chi_A + \chi_B$

(iv) $\chi_{A'} = 1 - \chi_A$

Suppose if $x \in A'$ then

$\chi_{A'}(x) = 1 - \chi_A(x)$

$= 1 - 0 = 1$ $\because x \notin A$

As we know that $A + A' = 1$.

for $x \in A$ then $x \notin A'$

then $\chi_A(x) = 0$

$\chi_{A'}(x) = 0 = 1 - \chi_A(x)$

$= 1 - 1 = 0$

$\chi_{A'}(x) = 1 - \chi_A(x)$

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THEOREM 2·10

Let 'f' be an extended real valued measurable function. Then 'f' is the pointwise limit of a sequence of simple functions. If $f \geq 0$, then the above sequence can be taken as monotonic increasing.

Proof:-

Let f be an non-negative simple function and for each $n \geq 1$ and $x \in X$ define

$$f_n(x) = \begin{cases} \frac{i-1}{2^n} & \text{if } \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \text{ for } i = 1, 2, 3 \ldots n2^n \\ n & \text{if } f(x) \geq n \end{cases}$$

$f(x) < n$

$n+1 = 4$ $\quad$ $n = 3$

$f(x) =$ $\quad$ $f_3(x)$

$i = 1.2.3.4.2^4$ $\quad$ $i = 1.2.3.2^3$

$= 24\,2^4$ $\quad$ $i = 8.2^3$

$f_4(x) = \frac{24\,2^4 - 1}{2^4}$ $\quad$ $f_3(x) = \frac{8.2^3 - 1}{2^3}$

Clearly each f_n is non negative simple function and

$f_{n+1}(x) \geq f_n(x)$ i.e

$\{f_n\}$ is monotonic increasing $\quad$ $23.9 \geq 7.87$

$\Rightarrow f_{n+1}(x) \geq f_n(x)$

Further if

$f(x) < n$

then

$0 \leq f(x) - f_n(x) \leq \frac{i}{2^n}$

If $f(x) = \infty$ Then $f_n(x) = n$ for every n

Hence $\lim_{n \to \infty} f_n(x) = f(x)$

BOREL MEASURABLE FUNCTIONS

BOREL SETS: A subset of $\mathbb{R}$ which can be written in the form of countable (union) or intersection of (open or closed) sets is called Borel set.

3·1 Borel Measurable function:

Let $(\mathbb{R}, \mathcal{B}, m)$ be the Borel measurable space. An extended real

valued function 'f' defined on a Borel set D is said to be Borel measurable if for each $\alpha \in \mathbb{R}$

$\{x \in D : f(x) > \alpha\} = f^{-1}(\alpha, \infty]$ is a Borel set

Definition 3.2

Let 'f' be an extended real valued function 'f' defined on a Borel set D. We say $f : D \to \bar{R}$ is a Borel measurable function if for each open set V in $\bar{R}$, $f^{-1}(V) \in \mathcal{B}$

Proposition 3.3:-

Prove that definition 3.1 and 3.2 are equivalent.

Proof:-

Definition $3.1 \Rightarrow 3.2$

Define $\Omega = \{E \subseteq \bar{R} : f^{-1}(E) \in \mathcal{B}\}$ be the collection of all subsets of $\bar{R}$ whose inverse is a Borel set.

First we will show that Ω is a σ-algebra.

Since $f : D \to \bar{R}$

$\Rightarrow f^{-1}(\bar{R}) = D \in \mathcal{B}$ so $\bar{R} \in \Omega$

Also $f^{-1}(\Phi) = f^{-1}(\bar{R})' = [f^{-1}(\bar{R})]'$

$= D' \in \mathcal{B}$ $\quad \because D \in \mathcal{B}$ ~~& $\mathcal{B}$ is σ-algebra~~

which implies $\phi \in \Omega$.

If $A \in \Omega$ then $f^{-1}(A) \in \mathcal{B}$ implies $f^{-1}(A') = [f^{-1}(A)]' \in \mathcal{B}$ (as $\mathcal{B}$ is a σ-algebra)

Thus $A' \in \Omega$

If $\{A_n\}$ is a sequence of sets in Ω, then $f^{-1}(A_n) \in \mathcal{B}$ for each n and hence

$f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} (f^{-1}(A_n)) \in \mathcal{B}$

which implies $f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) \in \mathcal{B}$

Thus $\bigcup_{n=1}^{\infty} A_n \in \Omega$

Thus Ω is a σ-algebra. ✓

Choose a real number α, such that $\alpha_n < \alpha$ and $\alpha_n \to \alpha$ as $n \to \infty$. By the definition 3·1 $f^{-1}(\alpha_n, \infty] \in \mathcal{B}$ so $(\alpha_n, \infty] \in \Omega$ for each n.

Also $[-\infty, \alpha) = \bigcup_{n=1}^{\infty} [-\infty, \alpha_n] = \bigcup_{n=1}^{\infty} (\alpha_n, \infty]' \in \Omega$

($\because \Omega$ is a σ-algebra) ✓

Consequently $(\alpha, \beta) = [-\infty, \beta) \cap (\alpha, \infty] \in \Omega$

As we know that every open set in $\bar{R}$ is a countable union of segments of the type (α, β), $[-\infty, \alpha)$, $(\alpha, \infty]$, which are members of Ω

Hence every open set V in $\bar{R}$ is in Ω

So By definition of Ω $f^{-1}(V) \in \mathcal{B}$ as required which gives definition 3·2

Definition 3·2 $\Rightarrow$ 3·1

By definition 3·2

For every open set V in $\bar{R}$, we have $f^{-1}(V) \in \mathcal{B}$. In particular $V = (\alpha, \infty]$ open then $f^{-1}(\alpha, \infty] = \{x : f(x) > \alpha\} \in \mathcal{B}$ which implies 'f' is a Borel Measurable.

Theorem 3·5: Imp A/2012

Let 'f' be an extended real valued function defined on a Borel set D. Then the following statements are equivalent:-

(i) f is Borel measurable

(ii) $\{x : f(x) \geq \alpha\}$ is a Borel set for all $\alpha \in \mathbb{R}$

(iii) $\{x : f(x) < \alpha\}$ is a Borel set for all $\alpha \in \mathbb{R}$

(iv) $\{x : f(x) \leq \alpha\}$ is a Borel set $\forall \alpha \in \mathbb{R}$

Moreover statement (i) ⇒ (iv) implies that

(v) $\{x : f(x) = \alpha\}$ is a Borel set for each $\alpha \in \bar{\mathbb{R}}$

Proof:-

(i) ⇒ (ii) :-

Suppose that 'f' is Borel Measurable function

i.e. for each α

$\{x : f(x) > \alpha\} = f^{-1}(\alpha, \infty] \in \mathcal{B}$

We need to show that

$\{x : f(x) \geq \alpha\}$ is a Borel set $\forall \alpha \in \bar{\mathbb{R}}$

We prove this by showing that

$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$

For this let $y \in \{x : f(x) \geq \alpha\}$

so that $f(y) \geq \alpha \quad \forall \alpha \in \bar{\mathbb{R}}$

In particular $f(y) > \alpha - 1/n$ for each $n \in N$

$\Rightarrow y \in \{x : f(x) > \alpha - 1/n\}$ for each $n \in N$

i.e $y \in \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$

Hence $\{x : f(x) \geq \alpha\} \subseteq \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$ —(i)

Now for converse inclusion

Let $t \in \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$

$\Rightarrow t \in \{x : f(x) > \alpha - 1/n\}$ for each $n \in N$

So $f(t) > \alpha - 1/n \quad \forall n \in N$

Thus $f(t) \geq \alpha \quad \forall \alpha \in \bar{\mathbb{R}}$

$\Rightarrow t \in \{x : f(x) \geq \alpha\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\} \subseteq \{x : f(x) \geq \alpha\}$ —(ii)

Hence (i) and (ii) implies

$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$

As $\{x : f(x) > \alpha - 1/n\}$ is Borel ~~measurable~~ set

$\because \mathcal{B}$ is σ-algebra so

$\{x : f(x) \geq \alpha\}$ being countable intersection

of Borel ~~Measurable~~ sets is Borel ~~Measurable~~ set

(ii) $\Rightarrow$ (iii) ✓

(ii) Implies that

$\{x : f(x) \geq \alpha\}$ is a Borel set $\forall \alpha \in \mathbb{R}$

we need to show that

$\{x : f(x) < \alpha\}$ is Borel set

We prove this by showing that

$\{x : f(x) < \alpha\} = D \setminus \{x : f(x) \geq \alpha\}$

For this let $y \in \{x : f(x) < \alpha\}$

$\Rightarrow f(y) < \alpha$

$\Rightarrow y \in D$ but $y \notin \{x : f(x) \geq \alpha\}$

$\Rightarrow y \in D \setminus \{x : f(x) \geq \alpha\}$ ...... (1)

$\Rightarrow \{x : f(x) < \alpha\} \subseteq D \setminus \{x : f(x) \geq \alpha\}$

for conversely inclusion

let $t \in D \setminus \{x : f(x) \geq \alpha\}$

$\Rightarrow t \in D$ but $t \notin \{x : f(x) \geq \alpha\}$

$\Rightarrow t \in \{x : f(x) < \alpha\}$

$\Rightarrow D \setminus \{x : f(x) \geq \alpha\} \subseteq \{x : f(x) < \alpha\}$ --- (2)

From (1) and (2)

$\{x : f(x) < \alpha\} = D \setminus \{x : f(x) \geq \alpha\}$

Since D is a Borel set & from (ii) $\{x : f(x) \geq \alpha\}$ is a Borel set so being difference of two Borel set

$\{x : f(x) < \alpha\}$ is a Borel set

(iii) $\Rightarrow$ (iv) ✓

(iii) implies that

$\{x : f(x) < \alpha\}$ is a Borel set

we need to show that

$\{x : f(x) \leq \alpha\}$ is a Borel set $\forall \alpha \in \mathbb{R}$

We prove this by showing that

$$\{x : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\}$$

For this let $y \in \{x : f(x) \leq \alpha\}$

$\Rightarrow f(y) \leq \alpha < \alpha + 1/n \quad \forall n \in N$

$\Rightarrow f(y) < \alpha + 1/n \quad \forall n \in N$

$\Rightarrow y \in \{x : f(x) < \alpha + 1/n\} \quad \forall n \in N$

Hence

$y \in \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\} \Rightarrow \{x : f(x) \leq \alpha\} \subseteq \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\}$ —(1)

Now for converse inclusion

let $t \in \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\}$

$\Rightarrow f(t) < \alpha + 1/n$ for each $n \in N$

$f(t) \leq \alpha \quad \forall \alpha \in \mathbb{R}$

$\Rightarrow t \in \{x : f(x) \leq \alpha\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\} \subseteq \{x : f(x) \leq \alpha\}$ —(2)

(1) & (2) implies

$$\{x : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\}$$

From (iii) $\{x : f(x) < \alpha + 1/n\}$ is a Borel set

Being countable intersections of Borel sets

$\{x : f(x) \leq \alpha\}$ is also a Borel set

(iv) $\Rightarrow$ (i)

(iv) $\Rightarrow \{x : f(x) \leq \alpha\}$ is a Borel set

To prove (i) i.e To show that 'f' is Borel measurable we need to show only that.

$\{x : f(x) > \alpha\}$ is a Borel set

we prove this by showing that

$\{x : f(x) > \alpha\} = D \setminus \{x : f(x) \leq \alpha\}$

Let $y \in \{x : f(x) > \alpha\}$

$\Rightarrow f(y) > \alpha$

$\Rightarrow y \in D$ but $y \notin \{x : f(x) \le \alpha\}$

$\Rightarrow y \in D \setminus \{x : f(x) \le \alpha\}$

$\Rightarrow \{x : f(x) > \alpha\} \subseteq D \setminus \{x : f(x) \le \alpha\}$ --- (1)

For converse inclusion.

Let $t \in D \setminus \{x : f(x) \le \alpha\}$

$\Rightarrow t \in D$ but

$t \notin \{x : f(x) \le \alpha\}$

$\Rightarrow t \in \{x : f(x) > \alpha\}$

$\Rightarrow D \setminus \{x : f(x) \le \alpha\} \subseteq \{x : f(x) > \alpha\}$ --- (2)

(1) and (2) implies that

$\{x : f(x) > \alpha\} = D \setminus \{x : f(x) \le \alpha\}$

As D and $\{x : f(x) \le \alpha\}$ are Borel sets

Being difference of two Borel sets

$\{x : f(x) > \alpha\}$ is a borel set

$\{x : f(x) > \alpha\} \in \mathcal{B}$

$\Rightarrow$ 'f' is Borel Measurable.

(V)

For any $\alpha \in \mathbb{R}$

$\{x : f(x) = \alpha\} = \{x : f(x) \ge \alpha\} \cap \{x : f(x) \le \alpha\}$

is a Borel set.

When $\alpha = \infty$

we have $\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) > n\}$ and

if

$\alpha = -\infty$

Then $\{x : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \le -n\}$

Consequently the set $\{x : f(x) = \alpha\}$ is a borel set for each extended real number α

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Theorem 3.7 Imp A/2011

Let $f: \mathbb{R} \to \mathbb{R}$ be a measurable function and $g: \mathbb{R} \to \mathbb{R}$ a Borel measurable function. Then $g \circ f$ is a measurable function. (In other words, Borel measurable function of a measurable function is measurable)

Define $\Omega = \{G \subseteq \mathbb{R} : f^{-1}(G) \in M\}$

Since $f^{-1}(\phi) = \phi$ and

$f^{-1}(\mathbb{R}) = \mathbb{R}$ are in M so ϕ and $\mathbb{R}$ belongs to Ω. ✓

If $A \in \Omega$ then $f^{-1}(A) \in M$

Now $f^{-1}(A') = [f^{-1}(A)]' \in M$ $\because M$ is σ-algebra

gives that $A' \in \Omega$

If $\{G_n\}$ is a sequence of sets in Ω then $f^{-1}(G_n) \in M$ for each n

Hence $f^{-1}\left(\bigcup_{n=1}^{\infty} G_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(G_n) \in M$

So

$\bigcup_{n=1}^{\infty} G_n \in \Omega$ Thus Ω is a σ-algebra on $\mathbb{R}$.

To show that $g \circ f$ is measurable, it is sufficient to prove that $(g \circ f)^{-1}(V)$ is a measurable set for any open set V in $\mathbb{R}$

Let V be an open set in $\mathbb{R}$. Since 'f' is measurable then $f^{-1}(V)$ is measurable

This implies $V \in \Omega$. ✓

Hence every open set of $\mathbb{R}$ is a member of Ω. Thus Ω contains the collection of $\mathcal{B}$ of all Borel sets of $\mathbb{R}$ i.e $\mathcal{B} \subseteq \Omega$.

($\because \mathcal{B}$ is collection of open intervals)

Now since g is Borel measurable

So $g^{-1}(V)$ is a Borel set and

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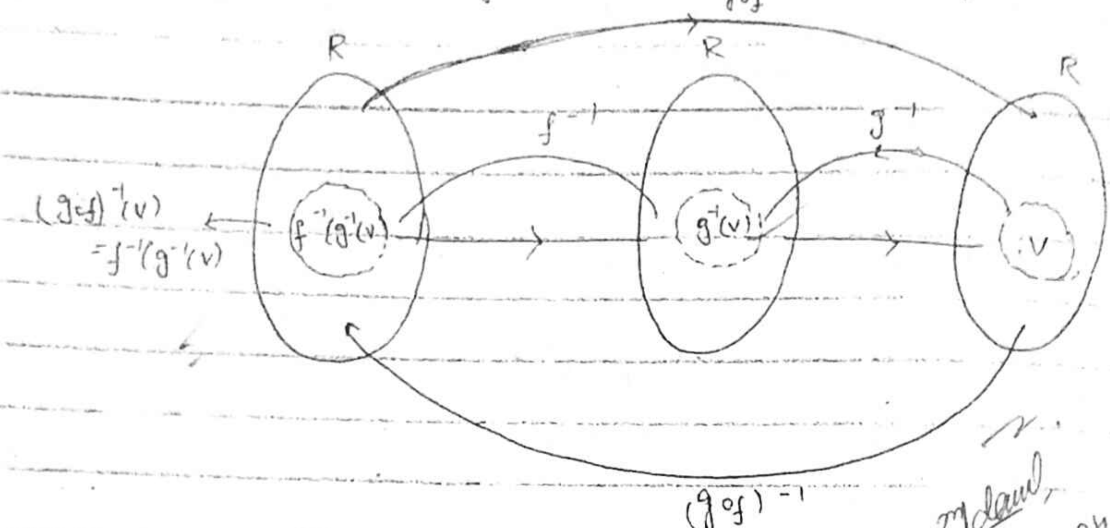
hence $g^{-1}(V) \in \mathcal{B} \subseteq \Omega$. Thus By definition of Ω, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is a measurable set. Hence $g \circ f$ is a measurable function



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Corollary 3.8:-

If $f: R \to R$ is measurable and $g: R \to R$ is continous function Then $g \circ f$ is a measurable function. Moreover if g is monotone and 'f' is measurable then $g \circ f$ is measurable.

THEOREM 3.9

If B is a Borel measurable subset of $\mathbb{R}$ Then $f^{-1}(B)$ is a measurable set if and only if 'f' is measurable

or If B is a Borel measurable subset of $\mathbb{R}$ then 'f' is measurable if and only if $f^{-1}(B)$ is measurable.

Proof:-

Let 'f' be a measurable function and B is a borel measurable subset of $\mathbb{R}$.

Define

$$\Omega = \{ G \subseteq \mathbb{R} : f^{-1}(G) \text{ is measurable i.e. } f^{-1}(G) \in \mathcal{M} \}$$

$(2+3i)^{-}$

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First we will show that Ω is σ-algebra
Since $f^{-1}(\phi) = \phi$ and $f^{-1}(R) = R$ are in $\mathcal{M}$
so ϕ and $\mathbb{R}$ belongs to Ω.

If $A \in \Omega$ then $f^{-1}(A) \in \mathcal{M}$
Now $f^{-1}(A') = [f^{-1}(A)]' \in \mathcal{M}$ $\because f^{-1}(A) \in \mathcal{M}$
which gives that $A' \in \Omega$

If $\{G_n\}$ is a sequence of sets in Ω then
$f^{-1}(G_n) \in \mathcal{M}$ for each n.
Hence $f^{-1}\left(\bigcup_{n=1}^{\infty} G_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(G_n) \in \mathcal{M}$
$\because f^{-1}(G_n) \in M$ for each n

So $\bigcup_{n=1}^{\infty} G_n \in \Omega$.
Thus Ω is a σ-algebra on $\mathbb{R}$
Suppose V is any open set in $\mathbb{R}$
Since 'f' is measurable so $f^{-1}(V)$ is measurable
which implies $V \in \Omega$. Hence every open set of $\mathbb{R}$
is a member of Ω. Thus Ω contains the
collection $\mathcal{B}$ of all Borel sets of $\mathbb{R}$ i.e. $\mathcal{B} \subseteq \Omega$
Thus by definition of Ω
$f^{-1}(B) \in \mathcal{M}$ i.e. $f^{-1}(B)$ is
measurable
Conversely suppose that $f^{-1}(B)$ is measurable
for any Borel set B in $\mathbb{R}$
In particular, if we take $B = (\alpha, \infty)$, $\alpha \in \mathbb{R}$
then $f^{-1}(B) = \{x : f(x) > \alpha\}$ is a measurable
set which implies 'f' is measurable.

$\because$ In each Borel set is open & so is measurable

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* Let $f : D \to \mathbb{R}$ is a Borel measurable function if for each open set V in $\mathbb{R}$, $f^{-1}(V) \in \mathcal{B}$

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Theorem 3.10 : Imp A/2013

Let 'f' be a Borel measurable function and B is a Borel set. Then $f^{-1}(B)$ is a Borel set.

Proof:→ Let f be a Borel measurable function & B be a borel set. Define $\Omega = \{ E \subseteq \mathbb{R} : f^{-1}(E) \in \mathcal{B} \}$

It is easy to show that Ω is a σ-algebra on $\mathbb{R}$.

* If V is an open set then Borel Measurability of f, $f^{-1}(V) \in \mathcal{B}$. (∵ Every open is Borel & so Borel meas.)

which implies $V \in \Omega$

Since B is the smallest σ-algebra which contains all open sets so it follows that $\mathcal{B} \in \Omega$. Hence any Borel set B is in Ω so $f^{-1}(B) \in \mathcal{B}$ by definition of Ω.

Remark:

For a function $f : D \to \mathbb{R}$ the following conditions are equivalent.

(a) 'f' is measurable,

(b) $f^{-1}(G)$ is measurable for each closed subset G of $\mathbb{R}$

(c) $f^{-1}(F)$ is measurable for each open subset F of $\mathbb{R}$.

(d) $f^{-1}(B)$ is measurable for each Borel subset B of $\mathbb{R}$.

NOTE:→ As we know that every Borel set is measurable. However converse is not true. i.e there may exist a measurable set which is not Borel.

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Theorem 3.11:

Let $f: D \to \overline{\mathbb{R}}$ and $g: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be Borel measurable functions. Then the composition $g \circ f: D \to \overline{\mathbb{R}}$ is Borel measurable.

Proof:-

Let $f: D \to \overline{\mathbb{R}}$ and $g: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be Borel measurable functions

Define $\Omega = \{ E \subseteq \overline{\mathbb{R}} : f^{-1}(E) \in \mathcal{B} \}$

It is easy to prove that Ω is a σ-algebra.

If V is an open set then by borel measurability of f, $f^{-1}(V) \in \mathcal{B}$

$\Rightarrow V \in \Omega$. Since $\mathcal{B}$ is smallest σ-algebra containing all the open sets so $\mathcal{B} \subseteq \Omega$. Hence any Borel set B in Ω and so $f^{-1}(B) \in \mathcal{B}$

Let $f: D \to \overline{\mathbb{R}}$ be an extended real valued function, f is measurable iff for any open set V in $\overline{\mathbb{R}}$ $f^{-1}(V) \in \mathcal{B}$

As 'g' is Borel measurable so so $g^{-1}(V) \in \mathcal{B}$ $(\subseteq \Omega)$ for any open set V in $\overline{\mathbb{R}}$.

Hence by definition of Ω,

$$f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \in \mathcal{B}$$

Thus $g \circ f$ is a Borel Measurable function

Theorem 3.12:-

Not every measurable set is a Borel set i.e ($\mathcal{B}$ is a proper subclass of $\mathcal{M}$)

Proof:-

Let $x \in [0,1]$

x in binary form is $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$ where $a_n = 0$ or 1

Let $f: [0,1] \to C$ be Cantor's function defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{2a_n}{3^n} \quad \forall\ x \in [0,1]$$

* For any Borel set B, f is measurable iff $f^{-1}(B)$ is measurable

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Let $x, y \in [0,1]$

The $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$ & $y = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$

Such that

$f(x) = f(y)$

$\sum_{n=1}^{\infty} \frac{2a_n}{3^n} = \sum_{n=1}^{\infty} \frac{2b_n}{3^n}$

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$

$\Rightarrow \sum_{n=1}^{\infty} \frac{a_n}{2^n} = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$

Hence

$x = y$

$\Rightarrow f$ is One One

For any $y \in C$ we have $y = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}$, $a_n = 0$ or 1

so there exist atleast one $x \in [0,1]$ such that

$f(x) = \sum_{n=1}^{\infty} \frac{2a_n}{3^n} = y$

Hence f is bijective i.e $C \simeq [0,1]$

Suppose P is any non measurable set in $[0,1]$

Then $A = f(P) \subseteq C$,

Since $A \subseteq C$, By Monotonicity

$0 \leq m^*(A) < m^*(C) = 0$ $\quad \because C \simeq [0,1]$, $m^*(C) = 0$

$\Rightarrow m^*(A) = 0$

$\Rightarrow A \in M$

If A were a Borel set Then $f^{-1}(A)$ would be Measurable set

But $f^{-1}(A) = f^{-1}(f(P)) = P$

which is a non measurable set (a Contradiction)

Consequently, The measurable set A is not a Borel set.

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